

# Extremal boundedness of a variational functional in point vortex mean field theory associated with probability measures

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## Abstract

We study a variational functional of Trudinger-Moser type associated with one-sided Borel probability measure. Its boundedness at the extremal parameter holds when the residual vanishing occurs. In the proof we use a variant of the Y.Y. Li estimate.

## 1 Introduction

The purpose of the present paper is to study the boundedness of a variational function concerning the mean field limit of many point vortices [21]. This limit takes the form

$$-\Delta v = \lambda \int_I \alpha \left( \frac{e^{\alpha v}}{\int_{\Omega} e^{\alpha v}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha) \quad \text{on } \Omega, \quad \int_{\Omega} v = 0, \quad (1.1)$$

where  $\Omega = (\Omega, g)$  is a compact and orientable Riemannian surface without boundary in dimension two,  $dx$  a volume element on  $\Omega$ , and  $|\Omega|$  the volume of  $\Omega$ :  $|\Omega| = \int_{\Omega} dx$ . The unknown variable  $v$  stands for the stream function of the fluid and  $\mathcal{P} = \mathcal{P}(d\alpha)$  is a Borel probability measure on  $I = [-1, 1]$ , representing a deterministic distribution of the circulation of vortices.

Single circulation is described by  $\mathcal{P} = \delta_{+1}$ . In this simplest case, equation (1.1) is sometimes called the mean field equation. Since Onsager's pioneering work of statistical mechanics on two-dimensional equilibrium turbulence [17], there are numerous mathematical and physical references in this case (see, for instance, [24, 26] and the references therein). Also, the other model  $\mathcal{P} = (1 - \tau)\delta_{-1} + \tau\delta_{+1}$ ,  $0 < \tau < 1$ , is concerned with signed vortices [9, 18]. Equation (1.1) is thus regarded as a generalization of these cases.

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There, the deterministic distribution of circulations is described by  $\mathcal{P}(d\alpha)$ . Several works are already devoted to equation (1.1), particularly, when  $\mathcal{P}$  is atomic [7, 10, 15, 16], i.e.,

$$\mathcal{P} = \sum_{i=1}^N b_i \delta_{\alpha_i}, \quad \alpha_i \in I, \quad b_i > 0, \quad \sum_{i=1}^N b_i = 1. \quad (1.2)$$

Actually, this model is equivalent to the Liouville system studied by [4, 6, 23]. We note that L. Onsager himself arrived at (1.1) for (1.2), see [8].

Model (1.1) is the Euler-Lagrange equation of the functional

$$J_\lambda(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \lambda \int_I \log \left( \int_{\Omega} e^{\alpha v} \right) \mathcal{P}(d\alpha), \quad v \in E,$$

where

$$E = \{v \in H^1(\Omega) \mid \int_{\Omega} v = 0\}.$$

Hence it may be the first step to clarify its boundedness to study (1.1) for general  $\mathcal{P}(d\alpha)$ . It is a kind of the Trudinger-Moser inequality,

$$\inf_{v \in E} J_\lambda(v) > -\infty. \quad (1.3)$$

In the atomic case of (1.2), the best constant of  $\lambda$  for (1.3) is known [23]. This result is originally described in the dual form of logarithmic HLS inequality (see [15] for (1.3)). Taking the limit, we can detect the extremal parameter  $\lambda = \bar{\lambda}$  for (1.3) to hold [20], that is,

$$\bar{\lambda} = \inf \left\{ \frac{8\pi \mathcal{P}(K_{\pm})}{\left( \int_{K_{\pm}} \alpha \mathcal{P}(d\alpha) \right)^2} \mid K_{\pm} \subset I_{\pm} \cap \text{supp } \mathcal{P} \right\}, \quad (1.4)$$

where

$$I_+ = [0, 1], \quad I_- = [-1, 0]$$

and

$$\text{supp } \mathcal{P} = \{\alpha \in I \mid \mathcal{P}(N) > 0 \text{ for any open neighborhood } N \text{ of } \alpha\}.$$

Thus we obtain

$$\begin{aligned} \lambda < \bar{\lambda} &\Rightarrow \inf_{v \in E} J_\lambda(v) > -\infty \\ \lambda > \bar{\lambda} &\Rightarrow \inf_{v \in E} J_\lambda(v) = -\infty. \end{aligned}$$

The inequality

$$\inf_{v \in E} J_{\bar{\lambda}}(v) > -\infty \quad (1.5)$$

however, is open, although (1.5) is the case if  $\mathcal{P}$  is atomic [23]. Here we take the fundamental assumption

$$\text{supp } \mathcal{P} \subset I_+ \quad (1.6)$$

and approach the problem as follows.

Namely, given  $\lambda_k \uparrow \bar{\lambda}$ , we have a minimizer  $v_k \in E$  of  $J_{\lambda}$ , that is,

$$\inf_{v \in E} J_{\lambda_k}(v) = J_{\lambda_k}(v_k).$$

If

$$\limsup_{k \rightarrow \infty} J_{\lambda_k}(v_k) > -\infty \quad (1.7)$$

we have

$$J_{\bar{\lambda}}(v) = \lim_{k \rightarrow \infty} J_{\lambda_k}(v) \geq \limsup_{k \rightarrow \infty} J_{\lambda_k}(v_k) > -\infty, \quad v \in E,$$

and hence (1.5) follows. In the case of

$$\sup_k \|v_k\|_{\infty} < +\infty,$$

inequality (1.7) is valid since  $v = v_k$  is a solution to (1.1) for  $\lambda = \lambda_k$ .

Assuming the contrary, we use a result of [15] concerning the non-compact solution sequence  $\{(\lambda_k, v_k)\}$  to (1.1). Regarding (1.6), we obtain

$$\begin{aligned} \mathcal{S} \equiv \{x_0 \in \Omega \mid \text{there exists } x_k \in \Omega \text{ such that} \\ x_k \rightarrow x_0 \text{ and } v_k(x_k) \rightarrow +\infty\} \neq \emptyset. \end{aligned}$$

This blowup set  $\mathcal{S}$  is finite and there is  $0 \leq s \in L^1(\Omega) \cap L_{loc}^{\infty}(\Omega \setminus \mathcal{S})$  such that

$$\nu_k \equiv \lambda_k \int_{I_+} \frac{\alpha e^{\alpha v_k}}{\int_{\Omega} e^{\alpha v_k}} \mathcal{P}(d\alpha) \xrightarrow{*} \nu \equiv s + \sum_{x_0 \in \mathcal{S}} n(x_0) \delta_{x_0} \quad \text{in } \mathcal{M}(\Omega) \quad (1.8)$$

with  $n(x_0) \geq 4\pi$  for each  $x_0 \in \mathcal{S}$ , where  $\delta_{x_0}$  denotes the Dirac measure centered at  $x_0$  and  $\mathcal{M}(\Omega)$  is the space of measures identified with the dual space of  $C(\Omega)$ . Under these preparations our main result is stated as follows.

**Theorem 1.** *Inequality (1.5) holds under the conditions (1.6) and  $s = 0$  in (1.8).*

Henceforth, we put

$$\alpha_{\min} = \inf_{\alpha \in \text{supp } \mathcal{P}} \alpha. \quad (1.9)$$

Here we have a note concerning the assumption made in the above theorem, that is,  $s = 0$  in (1.8), which we call the *residual vanishing*. This condition is actually satisfied under a suitable assumption on  $\mathcal{P}$ .

**Proposition 1** ([25] Theorem 3). *Residual vanishing,  $s = 0$ , occurs to the above  $\{v_k\}$  if  $\alpha_{\min} > 1/2$ .*

An immediate consequence is the following theorem.

**Theorem 2.** *We have (1.5) under  $\alpha_{\min} > 1/2$ .*

So far, there is no known inequality (1.5) for continuous  $\mathcal{P}(d\alpha)$  except for Theorem 2.

Residual vanishing implies the following property used in the proof of Theorem 1.

**Proposition 2** ([25] Lemma 3). *If the residual vanishing occurs to the above  $\{v_k\}$  then it holds that*

$$\sharp \mathcal{S} \leq 1, \quad \bar{\lambda} = \frac{8\pi}{\left(\int_{I_+} \alpha \mathcal{P}(d\alpha)\right)^2}. \quad (1.10)$$

Theorem 2 contains the classical case of the Trudinger-Moser inequality for  $\mathcal{P} = \delta_{+1}$ . We shall show a variant of Y.Y. Li's estimate [11], which is the key of the proof of Theorem 1. As we see later on, it takes the form that is weaker than the estimate shown for  $\mathcal{P} = \delta_{+1}$  in [11].

To state the result, let

$$w_{k,\alpha}(x) = \alpha v_k(x) - \log \int_{\Omega} e^{\alpha v_k}, \quad k \in \mathbf{N}, \quad \alpha \in I_+ \setminus \{0\}, \quad (1.11)$$

which satisfies

$$-\Delta w_{k,\alpha} = \alpha \lambda_k \int_{I_+} \beta \left( e^{w_{k,\beta}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\beta) \quad \text{on } \Omega, \quad \int_{\Omega} e^{w_{k,\alpha}} = 1. \quad (1.12)$$

Regarding (1.10), we put  $\mathcal{S} = \{x_0\}$ . There exists  $x_k \in \Omega$  such that

$$x_k \rightarrow x_0, \quad v_k(x_k) = \max_{\Omega} v_k, \quad w_{k,\alpha}(x_k) = \max_{\Omega} w_{k,\alpha}.$$

Here we take an isothermal chart  $(U_k, \Psi_k)$  satisfying

$$\Psi_k(x_k) = 0 \in \mathbf{R}^2, \quad g = e^{\xi_k(X)}(dX_1^2 + dX_2^2), \quad \xi_k(0) = 0.$$

Then it holds that

$$-\Delta_X w_{k,\alpha} = e^{\xi_k} \left( \alpha \lambda_k \int_{I_+} \beta \left( e^{w_{k,\beta}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\beta) \right) \quad \text{in } \Psi_k(U_k), \quad (1.13)$$

where

$$w_{k,\alpha}(X) = w_{k,\alpha} \circ \Psi_k^{-1}(X).$$

Henceforth, we shall write  $X$  by the same notation  $x$  for simplicity. Also, we do not distinguish any sequences with their subsequences. Under this agreement we have  $x_k = x_0 = 0$ . Moreover, there exists  $R_0 > 0$  such that  $B_{3R_0} \subset \subset \Psi_k(U_k)$  and 0 is the maximizer of  $w_{k,\alpha}$  in  $\bar{B}_{3R_0}$ .

The estimate is now stated in the following proposition.

**Proposition 3.** *Under the assumptions of Theorem 1 it holds that*

$$w_{k,\alpha}(x) - w_{k,\alpha}(0) = -\alpha (\gamma_0 + o(1)) \log(1 + e^{w_{k,1}(0)/2}|x|) + O(1) \quad (1.14)$$

as  $k \rightarrow \infty$  uniformly in  $x \in B_{R_0}$  and  $\alpha \in I_+ \setminus \{0\}$ , where

$$\gamma_0 = \frac{4}{\int_{I_+} \beta \mathcal{P}(d\beta)}.$$

To conclude this section, we shall describe a sketch of the proof of Theorem 1. The first step is the blowup analysis. We put

$$\tilde{w}_{k,\alpha}(x) = w_{k,\alpha}(\sigma_k x) + 2 \log \sigma_k, \quad \sigma_k = e^{-w_k(0)/2} \rightarrow 0, \quad \tilde{w}_k(x) = \tilde{w}_{k,1}(x),$$

and get

$$\begin{aligned} -\Delta \tilde{w}_{k,\alpha} &= \alpha(\tilde{f}_k - \tilde{\delta}_k e^{\tilde{\xi}_k}), \quad \tilde{w}_{k,\alpha} \leq \tilde{w}_{k,\alpha}(0) \leq \tilde{w}_k(0) = 0 \quad \text{in } B_{R_0/\sigma_k} \\ \int_{B_{R_0/\sigma_k}} e^{\tilde{w}_{k,\alpha} + \tilde{\xi}_k} &\leq 1, \quad \int_{B_{R_0/\sigma_k}} \tilde{f}_k \leq \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta) \end{aligned}$$

for each  $\alpha \in I_+ \setminus \{0\}$ , where

$$\tilde{f}_k = \lambda_k \int_{I_+} \beta e^{\tilde{w}_{k,\beta} + \tilde{\xi}_k} \mathcal{P}(d\beta), \quad \tilde{\delta}_k = \frac{\sigma_k^2 \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta)}{|\Omega|}, \quad \tilde{\xi}_k(x) = \xi(\sigma_k x).$$

The compactness argument assures the existence of  $\tilde{w} = \tilde{w}(x)$  and  $\tilde{f} = \tilde{f}(x)$  such that

$$\tilde{w}_k \rightarrow \tilde{w}, \quad \tilde{f}_k \rightarrow \tilde{f} \quad \text{in } C_{loc}^2(\mathbf{R}^2)$$

and

$$\begin{aligned} -\Delta \tilde{w} &= \tilde{f} \neq 0, \quad \tilde{w} \leq \tilde{w}(0) = 0, \quad 0 \leq \tilde{f} \leq \bar{\lambda} \int_{I_+} \beta \mathcal{P}(d\beta) \quad \text{in } \mathbf{R}^2 \\ \int_{\mathbf{R}^2} e^{\tilde{w}} &\leq 1, \quad \int_{\mathbf{R}^2} \tilde{f} \leq \bar{\lambda} \int_{I_+} \beta \mathcal{P}(d\beta). \end{aligned}$$

Next we focus on the quantity

$$\tilde{\gamma} = \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{f}.$$

Given a bounded open set  $\omega \subset \mathbf{R}^2$ , we have

$$\int_{I_+} \left( \int_{\omega} e^{\tilde{w}_{k,\beta} + \tilde{\xi}_k} dx \right) \mathcal{P}(d\beta) \leq 1,$$

and hence there exists  $\tilde{\zeta}^\omega = \tilde{\zeta}^\omega(d\beta) \in \mathcal{M}(I_+)$  such that

$$\left( \int_{\omega} e^{\tilde{w}_{k,\beta} + \tilde{\xi}_k} dx \right) \mathcal{P}(d\beta) \xrightarrow{*} \tilde{\zeta}^\omega(d\beta) \quad \text{in } \mathcal{M}(I_+).$$

For this limit measure, we can show the absolute continuity with respect to  $\mathcal{P}$ , equivalently, the existence of  $\tilde{\psi}^\omega \in L^1(I_+, \mathcal{P})$  such that  $0 \leq \tilde{\psi}^\omega \leq 1$   $\mathcal{P}$ -a.e. on  $I_+$  and

$$\tilde{\zeta}^\omega(\eta) = \int_{\eta} \tilde{\psi}^\omega(\beta) \mathcal{P}(d\beta)$$

for any Borel set  $\eta \subset I_+$ . Taking  $R_j \uparrow +\infty$  and putting  $\omega_j = B_{R_j}$ , we have  $\tilde{\zeta} \in \mathcal{M}(I_+)$  and  $\tilde{\psi} \in L^1(I_+, \mathcal{P})$  such that

$$\begin{aligned} 0 &\leq \tilde{\psi}(\beta) \leq 1, \quad \mathcal{P}\text{-a.e. } \beta \\ 0 &\leq \tilde{\psi}^{\omega_1}(\beta) \leq \tilde{\psi}^{\omega_2}(\beta) \leq \dots \rightarrow \tilde{\psi}(\beta), \quad \mathcal{P}\text{-a.e. } \beta \\ \tilde{\zeta}(\eta) &= \int_{\eta} \tilde{\psi}(\beta) \mathcal{P}(d\beta) \quad \text{for any Borel set } \eta \subset I_+ \end{aligned}$$

by the monotonicity of  $\tilde{\psi}^\omega$  with respect to  $\omega$ . Since it holds that

$$\bar{\lambda} \int_{I_+} \beta \tilde{\psi}^{\omega_j}(\beta) \mathcal{P}(d\beta) = \lim_{k \rightarrow \infty} \lambda_k \int_{I_+} \beta \left( \int_{\omega_j} e^{\tilde{w}_{k,\beta} + \tilde{\xi}_k} dx \right) \mathcal{P}(d\beta) = \int_{\omega_j} \tilde{f},$$

we have

$$\tilde{\gamma} = \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{f} = \frac{\bar{\lambda}}{2\pi} \int_{I_+} \beta \tilde{\psi}(\beta) \mathcal{P}(d\beta)$$

by the monotone convergence theorem. Furthermore, we use the Pohozaev identity and the behavior of  $\tilde{w}$  at infinity to obtain

$$-\pi \tilde{\gamma}^2 = -2\bar{\lambda} \int_{I_+} \tilde{\psi}(\beta) \mathcal{P}(d\beta). \quad (1.15)$$

More precisely, the following property holds.

**Proposition 4.** *It holds that*

$$\tilde{\gamma} = \frac{4}{\int_{I_+} \beta \mathcal{P}(d\beta)}, \quad (1.16)$$

in other words,

$$\tilde{\psi} = 1 \quad \mathcal{P}\text{-a.e. on } I_+. \quad (1.17)$$

Note that (1.16) and (1.17) are equivalent by (1.15) and (1.10). By virtue of Proposition 4, we can apply the method of [14] to prove Proposition 3. Finally, we use Proposition 3 and the another representation of  $J_{\lambda_k}(v_k)$  denoted by

$$\begin{aligned} J_{\lambda_k}(v_k) &= \frac{\lambda_k}{2} \left\{ \int_{I_+} (\bar{w}_{k,\alpha} + w_{k,\alpha}(0)) \mathcal{P}(d\alpha) \right. \\ &\quad \left. + \int_{I_+} \mathcal{P}(d\alpha) \int_{\Omega} (w_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{w_{k,\alpha}(x)} dx \right\} \end{aligned}$$

to show (1.7), where  $\bar{w}_{k,\alpha} = \frac{1}{|\Omega|} \int_{\Omega} w_{k,\alpha}$ .

This paper consists of five sections and Appendix. Sections 2 and 3 are devoted to the preliminary and the proof of Proposition 4, respectively. Then, we prove Proposition 3 in Section 4. The proof of Theorem 1 is provided in Section 5. An auxiliary lemma in Section 2 is shown in Appendix.

## 2 Preliminaries

We start with the following monotonicity properties.

**Lemma 2.1.** *For  $\alpha \in I_+$ , we have*

$$\frac{d}{d\alpha} w_{k,\alpha}(0) \geq 0 \quad (2.1)$$

$$\frac{d}{d\alpha} \int_{\Omega} e^{\alpha v_k} \geq 0. \quad (2.2)$$

*Proof.* We calculate

$$\begin{aligned} \frac{d}{d\alpha} w_{k,\alpha}(0) &= v_k(0) - \frac{\int_{\Omega} v_k e^{\alpha v_k}}{\int_{\Omega} e^{\alpha v_k}} \geq v_k(0) - \frac{\int_{\{v_k > 0\}} v_k e^{\alpha v_k}}{\int_{\Omega} e^{\alpha v_k}} \\ &\geq v_k(0) \left( 1 - \frac{\int_{\{v_k > 0\}} e^{\alpha v_k}}{\int_{\Omega} e^{\alpha v_k}} \right) \geq 0 \end{aligned}$$

for  $k$  and  $\alpha \in I_+$ , recalling that 0 is the maximizer of  $v_k$ , and

$$\frac{d}{d\alpha} \int_{\Omega} e^{\alpha v_k} = \int_{\Omega} v_k e^{\alpha v_k} = \int_{\Omega} v_k (e^{\alpha v_k} - 1) \geq 0$$

by using  $\int_{\Omega} v_k = 0$  and  $s(e^{\alpha s} - 1) \geq 0$  which is true for  $s \in \mathbf{R}$  and  $\alpha \geq 0$ .  $\square$

Henceforth, we put

$$w_k(x) = w_{k,1}(x).$$

It follows from (2.1) that

$$w_{k,1}(0) = \max_{\alpha \in I_+} w_{k,\alpha}(0). \quad (2.3)$$

The following lemma is the starting point of our blowup analysis.

**Lemma 2.2.** *For every  $\alpha \in I_+ \setminus \{0\}$ , it holds that*

$$w_{k,\alpha}(0) = \max_{B_{3R_0}} w_{k,\alpha} \rightarrow +\infty.$$

*Proof.* Since  $e^{w_{k,\alpha}(0)} = e^{\alpha v_k(0)} / \int_{\Omega} e^{\alpha v_k} \geq |\Omega|^{\alpha-1} e^{\alpha w_{k,1}(0)}$  for  $\alpha \in I_+ \setminus \{0\}$ , it suffices to show that  $w_k(0) = w_{k,1}(0) \rightarrow +\infty$ .

Residual vanishing and (2.2) imply  $\int_{\Omega} e^{v_k} \rightarrow +\infty$ , and then the local uniform boundedness of  $v_k$  in  $\Omega \setminus \{x_0\}$  shows that  $w_k \rightarrow -\infty$  locally uniformly in  $\Omega \setminus \{x_0\}$ . Hence if  $\lim_{k \rightarrow \infty} w_k(0) = \lim_{k \rightarrow \infty} \|w_k\|_{L^\infty(\Omega)} < +\infty$  then  $\lim_{k \rightarrow \infty} \int_{\Omega} e^{w_k} = 0$ , which contradicts  $\int_{\Omega} e^{w_k} = 1$ .  $\square$

We put

$$\tilde{w}_{k,\alpha}(x) = w_{k,\alpha}(\sigma_k x) + 2 \log \sigma_k, \quad \sigma_k = e^{-w_k(0)/2} \rightarrow 0, \quad \tilde{w}_k(x) = \tilde{w}_{k,1}(x).$$

The last notation is consistent under the agreement of  $w_k = w_{k,1}$ . For each  $\alpha \in I_+ \setminus \{0\}$  we have

$$-\Delta \tilde{w}_{k,\alpha} = \alpha(\tilde{f}_k - \tilde{\delta}_k e^{\tilde{\xi}_k}), \quad \tilde{w}_{k,\alpha} \leq \tilde{w}_{k,\alpha}(0) \leq \tilde{w}_k(0) = 0 \quad \text{in } B_{R_0/\sigma_k} \quad (2.4)$$

$$\int_{B_{R_0/\sigma_k}} e^{\tilde{w}_{k,\alpha} + \tilde{\xi}_k} \leq 1, \quad \int_{B_{R_0/\sigma_k}} \tilde{f}_k \leq \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta), \quad (2.5)$$

where

$$\tilde{f}_k = \lambda_k \int_{I_+} \beta e^{\tilde{w}_{k,\beta} + \tilde{\xi}_k} \mathcal{P}(d\beta), \quad \tilde{\delta}_k = \frac{\sigma_k^2 \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta)}{|\Omega|}, \quad \tilde{\xi}_k(x) = \xi(\sigma_k x). \quad (2.6)$$

We shall use a fundamental fact of which proof is provided in Appendix.

**Lemma 2.3.** *Given  $f \in L^1 \cap L^\infty(\mathbf{R}^2)$ , let*

$$z(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} f(y) \log \frac{|x-y|}{1+|y|} dy.$$

*Then, it holds that*

$$\lim_{|x| \rightarrow +\infty} \frac{z(x)}{\log |x|} = \gamma \equiv \frac{1}{2\pi} \int_{\mathbf{R}^2} f.$$

The following lemma is also classical (see [19] p. 130).

**Lemma 2.4.** *If  $\phi = \phi(x)$  is a harmonic function on the whole space  $\mathbf{R}^2$  such that*

$$\phi(x) \leq C_1(1 + \log |x|), \quad x \in \mathbf{R}^2 \setminus B_1$$

*then it is a constant function.*

Now we derive the limit of (2.4)-(2.5).

**Proposition 5.** *It holds that*

$$\tilde{w}_k \rightarrow \tilde{w}, \quad \tilde{f}_k \rightarrow \tilde{f} \quad \text{in } C_{loc}^2(\mathbf{R}^2) \quad (2.7)$$

for  $\tilde{w} = \tilde{w}(x)$  and  $\tilde{f} = \tilde{f}(x)$  satisfying

$$\begin{aligned} -\Delta \tilde{w} &= \tilde{f} \neq 0, \quad \tilde{w} \leq \tilde{w}(0) = 0, \quad 0 \leq \tilde{f} \leq \bar{\lambda} \int_{I_+} \beta \mathcal{P}(d\beta) \quad \text{in } \mathbf{R}^2 \\ \int_{\mathbf{R}^2} e^{\tilde{w}} &\leq 1, \quad \int_{\mathbf{R}^2} \tilde{f} \leq \bar{\lambda} \int_{I_+} \beta \mathcal{P}(d\beta). \end{aligned} \quad (2.8)$$

In addition, it holds that

$$\tilde{w}(x) \geq -\tilde{\gamma} \log(1 + |x|) + \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{f}(y) \log \frac{|y|}{1+|y|} dy \quad (2.9)$$

for any  $x \in \mathbf{R}^2$ , where

$$\tilde{\gamma} = \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{f}. \quad (2.10)$$

*Proof.* We have

$$\tilde{w}_{k,\beta}(x) = \beta \tilde{w}_k(x) + (w_{k,\beta}(0) - w_k(0)) \quad (2.11)$$

for any  $\beta \in I_+ \setminus \{0\}$ , and also

$$\tilde{w}_k \leq \tilde{w}_k(0) = 0, \quad w_{k,\beta}(0) \leq w_k(0), \quad \beta \in I_+ \setminus \{0\} \quad (2.12)$$

by (2.3). Hence  $\tilde{f}_k = \tilde{f}_k(x)$  satisfies

$$0 \leq \tilde{f}_k(x) \leq \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta) \cdot \sup_{B_{R_0}} e^{\xi_k} \quad \text{in } B_{R_0/\sigma_k}. \quad (2.13)$$



Fix  $L > 0$  and decompose  $\tilde{w}_k$ ,  $k \gg 1$ , as  $\tilde{w}_k = \tilde{w}_{1,k} + \tilde{w}_{2,k} + \tilde{w}_{3,k}$ , where  $\tilde{w}_{k,j}$ ,  $j = 1, 2, 3$ , are the solutions to

$$\begin{aligned} -\Delta \tilde{w}_{1,k} &= \tilde{f}_k \geq 0 \quad \text{in } B_L, & \tilde{w}_{1,k} &= 1 \quad \text{on } \partial B_L \\ -\Delta \tilde{w}_{2,k} &= -\tilde{\delta}_k e^{\tilde{\xi}_k} \leq 0 \quad \text{in } B_L, & \tilde{w}_{2,k} &= 0 \quad \text{on } \partial B_L \\ -\Delta \tilde{w}_{3,k} &= 0 \quad \text{in } B_L, & \tilde{w}_{3,k} &= \tilde{w}_k - 1 \quad \text{on } \partial B_L. \end{aligned}$$

First, there exists  $C_{2,L} > 0$  such that

$$1 \leq \tilde{w}_{1,k} \leq C_{2,L} \quad \text{on } \overline{B_L}.$$

Next it follows from  $\tilde{\delta}_k \rightarrow 0$  that

$$-\frac{1}{2} \leq \tilde{w}_{2,k} \leq 0 \quad \text{on } \overline{B_L}.$$

Finally, we have

$$\tilde{w}_{3,k} \leq -1 \quad \text{on } \overline{B_L}$$

by  $\tilde{w}_k \leq 0$ . Hence  $\tilde{w}_{3,k} = \tilde{w}_{3,k}(x)$  is a negative harmonic function in  $B_L$ . Then the Harnack inequality yields  $C_{3,L} > 0$  such that

$$\tilde{w}_{3,k} \geq -C_{3,L} \quad \text{in } \overline{B_{L/2}}.$$

We thus end up with

$$\frac{1}{2} - C_{3,L} \leq \tilde{w}_k \leq \tilde{w}_k(0) = 0 \quad \text{in } B_{L/2}, \quad (2.14)$$

and then the standard compactness argument assures the limit (2.7)-(2.8) thanks to (2.13), (2.14),  $\xi_k(0) = 0$  and  $\tilde{\delta}_k \rightarrow 0$ .

If  $\tilde{f} \equiv 0$  then

$$-\Delta \tilde{w} = 0, \quad \tilde{w} \leq \tilde{w}(0) = 0 \quad \text{in } \mathbf{R}^2, \quad \int_{\mathbf{R}^2} e^{\tilde{w}} \leq 1,$$

which is impossible by the Liouville theorem, and hence  $\tilde{f} \not\equiv 0$ .

Since  $\tilde{f} \in L^1 \cap L^\infty(\mathbf{R}^2)$ , the function

$$\tilde{z}(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{f}(y) \log \frac{|x-y|}{1+|y|} dy \quad (2.15)$$

is well-defined, and satisfies

$$\frac{\tilde{z}(x)}{\log |x|} \rightarrow \tilde{\gamma} = \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{f} \quad \text{as } |x| \rightarrow +\infty \quad (2.16)$$

by Lemma 2.3. Also (2.16) implies

$$\begin{aligned} -\Delta \tilde{w} &= \tilde{f}, & -\Delta \tilde{z} &= -\tilde{f}, & \tilde{w} &\leq \tilde{w}(0) = 0 \quad \text{in } \mathbf{R}^2 \\ \tilde{z}(x) &\leq (\tilde{\gamma} + 1) \log |x|, & x &\in \mathbf{R}^2 \setminus B_R \end{aligned}$$

for some  $R > 0$  by (2.16). Hence we obtain  $\tilde{u} \equiv \tilde{w} + \tilde{z} \equiv \text{constant}$  by Lemma 2.4. Since  $\tilde{w}(0) = 0$  it holds that

$$\tilde{w}(x) = -\tilde{z}(x) + \tilde{z}(0). \quad (2.17)$$

Now we note

$$\begin{aligned}\tilde{z}(x) &\leq \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{f}(y) \log \frac{|x|+|y|}{1+|y|} dy \\ &\leq \log(1+|x|) \cdot \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{f} = \tilde{\gamma} \log(1+|x|)\end{aligned}$$

by  $\tilde{f} \geq 0$ . Hence,  $\tilde{w}(x) \geq -\tilde{\gamma} \log(1+|x|) + \tilde{z}(0)$ , and the proof is complete.  $\square$

To study  $\tilde{\gamma}$  in (2.10), let

$$\mathcal{B} = \{\beta \in \text{supp } \mathcal{P} \mid \limsup_{k \rightarrow \infty} (w_{k,\beta}(0) - w_k(0)) > -\infty\}. \quad (2.18)$$

From the proof of Proposition 5, it follows that if  $\mathcal{P}(\mathcal{B}) = 0$  then  $\tilde{f} \equiv 0$ , a contradiction. Hence  $\mathcal{P}(\mathcal{B}) > 0$ , and the value

$$\beta_{\inf} = \inf_{\beta \in \mathcal{B}} \beta \quad (2.19)$$

is well-defined. Then we find

$$\mathcal{B} = I_{\inf} \cap \text{supp } \mathcal{P} \quad (2.20)$$

by the monotonicity (2.1), where

$$I_{\inf} = \begin{cases} [\beta_{\inf}, 1] & \text{if } \beta_{\inf} \in \mathcal{B}, \\ (\beta_{\inf}, 1] & \text{if } \beta_{\inf} \notin \mathcal{B}. \end{cases}$$

**Lemma 2.5.**  $\beta_{\inf} \tilde{\gamma} > 2$ .

*Proof.* By the definition, every  $\beta \in \mathcal{B}$  admits a subsequence such that  $\tilde{w}_{k,\beta}(0) = w_{k,\beta}(0) - w_k(0) = O(1)$ . We recall that  $\tilde{w}_{k,\beta}$  satisfies (2.4) for  $\alpha = \beta$ , i.e.,

$$-\Delta \tilde{w}_{k,\beta} = \beta(-\Delta \tilde{w}_k) = \beta(\tilde{f}_k - \tilde{\delta}_k e^{\tilde{\xi}_k}).$$

From the argument developed for the proof of (2.7)-(2.9), we have  $\tilde{w}_\beta = \tilde{w}_\beta(x) \in C^2(\mathbf{R}^2)$  such that

$$\tilde{w}_{k,\beta} \rightarrow \tilde{w}_\beta \quad \text{in } C_{loc}^2(\mathbf{R}^2). \quad (2.21)$$

The limit  $\tilde{w}_\beta$  satisfies

$$-\Delta \tilde{w}_\beta = \beta \tilde{f}, \quad \tilde{w}_\beta \leq \tilde{w}_\beta(0) \leq 0 \quad \text{in } \mathbf{R}^2, \quad \int_{\mathbf{R}^2} e^{\tilde{w}_\beta} \leq 1$$

and

$$\tilde{w}_\beta(x) \geq -\beta \tilde{\gamma} \log(1+|x|) + \frac{\beta}{2\pi} \int_{\mathbf{R}^2} \tilde{f}(y) \log \frac{|y|}{1+|y|} dy \quad (2.22)$$

with  $\tilde{f} = \tilde{f}(x)$  given in Proposition 5.

If  $\beta_{\inf} \in \mathcal{B}$ , we take  $\beta = \beta_{\inf}$ . Since  $\tilde{f} \in L^1 \cap L^\infty(\mathbf{R}^2)$  and  $\int_{\mathbf{R}^2} e^{\tilde{w}_\beta} < +\infty$ , we obtain the lemma by (2.22).

If  $\beta_{\inf} \notin \mathcal{B}$ , we take  $\beta_j \in \mathcal{B}$  in  $\beta_j \downarrow \beta_{\inf}$  and apply (2.22) for  $\beta = \beta_j$ . Since

$$\frac{\beta_j}{2\pi} \int_{\mathbf{R}^2} \tilde{f}(y) \log \frac{|y|}{1+|y|} dy = O(1), \quad \int_{\mathbf{R}^2} e^{\tilde{w}_{\beta_j}} \leq 1$$

there is  $\varepsilon_0 > 0$  independent of  $j$  such that  $\beta_j \tilde{\gamma} \geq 2 + \varepsilon_0$ , and then we obtain the lemma.  $\square$

Given a bounded open set  $\omega \subset \mathbf{R}^2$ , we have

$$\int_{I_+} \left( \int_{\omega} e^{\tilde{w}_{k,\beta} + \tilde{\xi}_k} dx \right) \mathcal{P}(d\beta) \leq 1.$$

Hence it holds that

$$\left( \int_{\omega} e^{\tilde{w}_{k,\beta} + \tilde{\xi}_k} dx \right) \mathcal{P}(d\beta) \xrightarrow{*} \tilde{\zeta}^{\omega}(d\beta) \quad \text{in } \mathcal{M}(I_+). \quad (2.23)$$

Now we shall show that the limit measure  $\tilde{\zeta}^{\omega} = \tilde{\zeta}^{\omega}(d\beta) \in \mathcal{M}(I_+)$  is absolutely continuous with respect to  $\mathcal{P}$ .

**Lemma 2.6.** *There exists  $\tilde{\psi}^{\omega} \in L^1(I_+, \mathcal{P})$  such that  $0 \leq \tilde{\psi}^{\omega} \leq 1$   $\mathcal{P}$ -a.e. on  $I_+$  and*

$$\tilde{\zeta}^{\omega}(\eta) = \int_{\eta} \tilde{\psi}^{\omega}(\beta) \mathcal{P}(d\beta)$$

for any Borel set  $\eta \subset I_+$ .

*Proof.* Let  $\eta \subset I_+$  be a Borel set and  $\varepsilon > 0$ . Then each compact set  $K \subset \eta$  admits an open set  $J \subset I_+$  such that

$$K \subset \eta \subset J, \quad \mathcal{P}(J) \leq \varepsilon + \mathcal{P}(K).$$

Now we take  $\varphi \in C(I_+)$  satisfying

$$\varphi = 1 \quad \text{on } K, \quad 0 \leq \varphi \leq 1 \quad \text{on } I_+, \quad \text{supp } \varphi \subset J.$$

Then (2.23) implies

$$\begin{aligned} \tilde{\zeta}^{\omega}(K) &= \int_K \tilde{\zeta}^{\omega}(d\beta) \leq \int_{I_+} \varphi(\beta) \tilde{\zeta}^{\omega}(d\beta) \\ &= \lim_{k \rightarrow \infty} \int_{I_+} \varphi(\beta) \left( \int_{\omega} e^{\tilde{w}_{k,\beta} + \tilde{\xi}_k} dx \right) \mathcal{P}(d\beta) \leq \int_{I_+} \varphi(\beta) \mathcal{P}(d\beta) \\ &\leq \int_J \mathcal{P}(d\beta) = \mathcal{P}(J) \leq \varepsilon + \mathcal{P}(\eta), \end{aligned}$$

and therefore

$$0 \leq \tilde{\zeta}^{\omega}(\eta) = \sup\{\tilde{\zeta}^{\omega}(K) \mid K \subset \eta : \text{compact}\} \leq \varepsilon + \mathcal{P}(\eta).$$

This shows the absolute continuity of  $\tilde{\zeta}^{\omega}$  with respect to  $\mathcal{P}$ .  $\square$

We take  $R_j \uparrow +\infty$  and put  $\omega_j = B_{R_j}$ . From the monotonicity of  $\tilde{\psi}^{\omega}$  with respect to  $\omega$ , there exist  $\tilde{\zeta} \in \mathcal{M}(I_+)$  and  $\tilde{\psi} \in L^1(I_+, \mathcal{P})$  such that

$$\begin{aligned} 0 &\leq \tilde{\psi}(\beta) \leq 1, \quad \mathcal{P}\text{-a.e. } \beta \\ 0 &\leq \tilde{\psi}^{\omega_1}(\beta) \leq \tilde{\psi}^{\omega_2}(\beta) \leq \dots \rightarrow \tilde{\psi}(\beta), \quad \mathcal{P}\text{-a.e. } \beta \\ \tilde{\zeta}(\eta) &= \int_{\eta} \tilde{\psi}(\beta) \mathcal{P}(d\beta) \quad \text{for any Borel set } \eta \subset I_+. \end{aligned}$$

First, (2.7) implies

$$\bar{\lambda} \int_{I_+} \beta \tilde{\psi}^{\omega_j}(\beta) \mathcal{P}(d\beta) = \lim_{k \rightarrow \infty} \lambda_k \int_{I_+} \beta \left( \int_{\omega_j} e^{w_{k,\beta} + \tilde{\xi}_k} dx \right) \mathcal{P}(d\beta) = \int_{\omega_j} \tilde{f}.$$

Then we obtain

$$\tilde{\gamma} = \frac{1}{2\pi} \int_{\mathbf{R}^2} \tilde{f} = \frac{\bar{\lambda}}{2\pi} \int_{I_+} \beta \tilde{\psi}(\beta) \mathcal{P}(d\beta) \quad (2.24)$$

by the monotone convergence theorem.

Similarly to [5], on the other hand, we have the following lemma, where  $(r, \theta)$  denotes the polar coordinate in  $\mathbf{R}^2$ .

**Lemma 2.7.** *We have*

$$\lim_{r \rightarrow +\infty} r \tilde{w}_r = -\tilde{\gamma}, \quad \lim_{r \rightarrow +\infty} \tilde{w}_\theta = 0$$

uniformly in  $\theta$ .

*Proof.* From (2.15) and (2.17), it follows that

$$\begin{aligned} r \tilde{w}_r(x) &= -\tilde{\gamma} - \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{y \cdot (x - y)}{|x - y|^2} \tilde{f}(y) dy \\ \tilde{w}_\theta(x) &= \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{\bar{y} \cdot (x - y)}{|x - y|^2} \tilde{f}(y) dy, \quad \bar{y} = (y_2, -y_1). \end{aligned}$$

Hence it suffices to show

$$\lim_{|x| \rightarrow +\infty} I_1(x) = \lim_{|x| \rightarrow +\infty} I_2(x) = 0,$$

where

$$I_1(x) = \int_{|x-y| > |x|/2} \frac{|y|}{|x-y|} \tilde{f}(y) dy, \quad I_2(x) = \int_{|x-y| \leq |x|/2} \frac{|y|}{|x-y|} \tilde{f}(y) dy.$$

Since  $\tilde{f} \in L^1(\mathbf{R}^2)$ , we have  $\lim_{|x| \rightarrow +\infty} I_1(x) = 0$  by the dominated convergence theorem.

Next, (2.7) implies

$$\begin{aligned} I_2(x) &= \lim_{k \rightarrow \infty} \int_{|x-y| \leq |x|/2} \frac{|y|}{|x-y|} \left( \lambda_k \int_{I_+} \beta e^{\tilde{w}_{k,\beta}(y) + \tilde{\xi}_k(y)} \mathcal{P}(d\beta) \right) dy \\ &= \bar{\lambda} \lim_{k \rightarrow \infty} \int_{[\beta_{\inf}, 1]} \beta \left( \int_{|x-y| \leq |x|/2} \frac{|y|}{|x-y|} e^{\tilde{w}_{k,\beta}(y) + \tilde{\xi}_k(y)} dy \right) \mathcal{P}(d\beta), \end{aligned}$$

recalling (2.18)-(2.19). Now we use (2.11)-(2.12) and (2.7) with (2.17), to confirm

$$\tilde{w}_{k,\beta}(x) \leq \beta \tilde{w}_k(x) = \beta(-\tilde{z}(x) + \tilde{z}(0)) + o(1)$$

as  $k \rightarrow \infty$ , locally uniformly in  $x \in \mathbf{R}^2$ . Hence it holds that

$$0 \leq I_2(x) \leq C_4 \int_{|x-y| \leq |x|/2} \frac{|y|}{|x-y|} \cdot \int_{[\beta_{\inf}, 1]} e^{-\beta \tilde{z}(y)} P(d\beta) dy.$$

Then (2.16) and Lemma 2.5 imply

$$0 \leq I_2(x) \leq C_5 |x|^{-(1+\varepsilon_0)} \int_{|x-y| \leq |x|/2} \frac{dy}{|x-y|} \leq C_6 |x|^{-\varepsilon_0}$$

with some  $\varepsilon_0 > 0$ , where we have used

$$|x-y| \leq \frac{|x|}{2} \Rightarrow \frac{1}{2}|x| \leq |y| \leq \frac{3}{2}|x|.$$

Hence  $\lim_{|x| \rightarrow +\infty} I_2(x) = 0$  follows.  $\square$

The Pohozaev identity

$$\begin{aligned} R \int_{\partial B_R} \frac{1}{2} |\nabla u|^2 - u_r^2 \, ds &= R \int_{\partial B_R} A(x) F(u) \, ds \\ &\quad - \int_{B_R} 2A(x) F(u) + F(u)(x \cdot \nabla A(x)) \, dx \end{aligned} \quad (2.25)$$

is valid to  $u = u(x) \in C^2(\overline{B_R})$  satisfying

$$-\Delta u = A(x) F'(u) \quad \text{in } B_R, \quad (2.26)$$

where  $F \in C^1(\mathbf{R})$ ,  $A \in C^1(\overline{B_R})$ , and  $ds$  denotes the surface element on the boundary. By this identity and Lemma 2.7 we obtain the following fact.

**Lemma 2.8.** *It holds that*

$$\int_{I_+} \tilde{\psi}(\beta) \mathcal{P}(d\beta) = \left( \int_{I_+} \phi_0(\beta) \tilde{\psi}(\beta) \mathcal{P}(d\beta) \right)^2, \quad (2.27)$$

where

$$\phi_0(\beta) = \frac{\beta}{\int_{I_+} \alpha \mathcal{P}(d\alpha)}. \quad (2.28)$$

*Proof.* We apply (2.25) for (2.26) to (2.4),  $\alpha = 1$ , where  $u = \tilde{w}_k$  and

$$F(\tilde{w}_k) = \lambda_k \int_{I_+} e^{\tilde{w}_k, \beta} \mathcal{P}(d\beta) - \tilde{\delta}_k \tilde{w}_k, \quad A(x) = e^{\tilde{\xi}_k(x)} = e^{\xi_k(\sigma_k x)}.$$

It follows that

$$\begin{aligned} R \int_{\partial B_R} \frac{1}{2} |\nabla \tilde{w}_k|^2 - (\tilde{w}_k)_r^2 \, ds &= -2\lambda_k \int_{I_+} \left( \int_{B_R} e^{\tilde{w}_k, \beta + \tilde{\xi}_k} dx \right) \mathcal{P}(d\beta) \\ &\quad + R\lambda_k \int_{I_+} \left( \int_{\partial B_R} e^{\tilde{w}_k, \beta + \tilde{\xi}_k} ds \right) \mathcal{P}(d\beta) - R\tilde{\delta}_k \int_{\partial B_R} \tilde{w}_k e^{\tilde{\xi}_k} ds \\ &\quad - \sigma_k \cdot \lambda_k \int_{I_+} \left( \int_{B_R} e^{\tilde{w}_k, \beta + \tilde{\xi}_k} (x \cdot \nabla \xi_k(\sigma_k x)) dx \right) \mathcal{P}(d\beta) \\ &\quad + \tilde{\delta}_k \int_{B_R} 2\tilde{w}_k e^{\tilde{\xi}_k} + \sigma_k \tilde{w}_k e^{\tilde{\xi}_k} (x \cdot \nabla \xi_k(\sigma_k x)) \, dx \end{aligned} \quad (2.29)$$

for  $\tilde{\delta}_k$  defined by (2.6).

For every  $R > 0$ , the last three terms on the right-hand side of (2.29) vanish as  $k \rightarrow \infty$ , because of  $\tilde{\delta}_k \rightarrow 0$  and  $\sigma_k \rightarrow 0$ . For the second term we argue similarly as in the proof of Lemma 2.7, while the conclusion of Lemma 2.7 is applicable to the left-hand side on (2.29). We thus end up with

$$-\pi\tilde{\gamma}^2 = -2\bar{\lambda} \int_{I_+} \tilde{\psi}(\beta) \mathcal{P}(d\beta) \quad (2.30)$$

by sending  $k \rightarrow \infty$  and then  $R \rightarrow +\infty$ .

Combining (2.30), (2.24), and the value  $\bar{\lambda}$  given in (1.10), we obtain (2.27)-(2.28).  $\square$

Let

$$\begin{aligned} \mathcal{I}^0(\psi) &= \int_{I_+} \phi_0(\beta) \psi(\beta) \mathcal{P}(d\beta) \\ \mathcal{C}_d &= \{\psi \mid 0 \leq \psi(\beta) \leq 1 \text{ } \mathcal{P}\text{-a.e. on } I_+ \text{ and } \int_{I_+} \psi(\beta) \mathcal{P}(d\beta) = d\} \end{aligned}$$

and  $\chi_A$  be the characteristic function of the set  $A$ . The following lemma is a variant of the result of [13].

**Lemma 2.9.** *For each  $0 < d \leq 1$ , the value  $\sup_{\psi \in \mathcal{C}_d} \mathcal{I}^0(\psi)$  is attained by*

$$\psi_d(\beta) = \chi_{\{\phi_0 > s_d\}}(\beta) + c_d \chi_{\{\phi_0 = s_d\}}(\beta) \quad (2.31)$$

with  $s_d$  and  $c_d$  defined by

$$\begin{aligned} s_d &= \inf\{t \mid \mathcal{P}(\{\phi_0 > t\}) \leq d\} \\ c_d \mathcal{P}(\{\phi_0 = s_d\}) &= d - \mathcal{P}(\{\phi_0 > s_d\}), \quad 0 \leq c_d \leq 1. \end{aligned} \quad (2.32)$$

Furthermore, the maximizer is unique in the sense that  $\psi_m = \psi_d$   $\mathcal{P}$ -a.e. on  $I_+$  for any maximizer  $\psi_m \in \mathcal{C}_d$ .

*Proof.* Fix  $0 < d \leq 1$ . Given  $\psi \in \mathcal{C}_d$ , we compute

$$\begin{aligned} \int_{I_+} \phi_0(\psi_d - \psi) \mathcal{P}(d\beta) &= \int_{\{\phi_0 > s_d\}} \phi_0(\psi_d - \psi) \mathcal{P}(d\beta) \\ &\quad + s_d \int_{\{\phi_0 = s_d\}} (\psi_d - \psi) \mathcal{P}(d\beta) - \int_{\{\phi_0 < s_d\}} \phi_0 \psi \mathcal{P}(d\beta) \\ &\geq s_d \int_{\{\phi_0 > s_d\}} (\psi_d - \psi) \mathcal{P}(d\beta) \\ &\quad + s_d \int_{\{\phi_0 = s_d\}} (\psi_d - \psi) \mathcal{P}(d\beta) - \int_{\{\phi_0 < s_d\}} \phi_0 \psi \mathcal{P}(d\beta) \end{aligned} \quad (2.33)$$

$$\begin{aligned} &\geq s_d \left( \int_{\{\phi_0 > s_d\}} (\psi_d - \psi) \mathcal{P}(d\beta) \right. \\ &\quad \left. + \int_{\{\phi_0 = s_d\}} (\psi_d - \psi) \mathcal{P}(d\beta) - \int_{\{\phi_0 < s_d\}} \psi \mathcal{P}(d\beta) \right) \\ &= s_d \int_{I_+} (\psi_d - \psi) \mathcal{P}(d\beta) = 0, \end{aligned} \quad (2.34)$$

which means that  $\psi_d$  is the maximizer.

The equalities hold in (2.33) and (2.34) if and only if  $\psi$  is the maximizer, and so we shall derive the conditions that the former is true. The first condition is that  $(\phi_0 - s_d)(\psi_d - \psi) = 0$   $\mathcal{P}$ -a.e. on  $\{\phi_0 > s_d\}$ , so that

$$\psi = \psi_d \quad \mathcal{P}\text{-a.e. on } \{\phi_0 > s_d\} \quad (2.35)$$

by the monotonicity of  $\phi_0$  and  $\psi_d \geq \psi$  on  $\{\phi_0 > s_d\}$ . The second one is that  $(s_d - \phi_0)\psi = 0$   $\mathcal{P}$ -a.e. on  $\{\phi_0 < s_d\}$ , or

$$\psi = 0 \quad \mathcal{P}\text{-a.e. on } \{\phi_0 < s_d\} \quad (2.36)$$

by the monotonicity of  $\phi_0$  and  $\psi \geq 0$ . The uniqueness follows from (2.35)-(2.36) and  $\psi_d, \psi \in \mathcal{C}_d$ .  $\square$

### 3 Proof of Proposition 4

For the purpose, we assume the contrary, that is,  $\tilde{\psi} \in \mathcal{C}_d$  for some  $0 < d < 1$ . We shall prove Proposition 4 by contradiction.

Since  $\tilde{\psi} = \tilde{\psi}(\beta)$  satisfies (2.27), it holds that

$$d = \int_{I_+} \tilde{\psi}(\beta) \mathcal{P}(d\beta) = \left( \int_{I_+} \phi_0(\beta) \tilde{\psi}(\beta) \mathcal{P}(d\beta) \right)^2.$$

Lemma 2.9 and (2.28) yield

$$d = \mathcal{P}(\{\phi_0 > s_d\}) + c_d \mathcal{P}(\{\phi_0 = s_d\}) \leq \left( \frac{\int_{I_+} \psi_d(\beta) \beta \mathcal{P}(d\beta)}{\int_{I_+} \beta \mathcal{P}(d\beta)} \right)^2 \quad (3.1)$$

for  $\psi_d = \psi_d(\beta)$  defined by (2.31)-(2.32). By the monotonicity of  $\phi_0 = \phi_0(\beta)$ , there exists the unique element  $\beta_d \in I_+$  such that

$$\phi_0(\beta_d) = s_d,$$

and then (3.1) reads

$$d = \mathcal{P}((\beta_d, 1]) + c_d \mathcal{P}(\{\beta_d\}) \leq \left( \frac{\int_{(\beta_d, 1]} \beta \mathcal{P}(d\beta) + c_d \beta_d \mathcal{P}(\{\beta_d\})}{\int_{I_+} \beta \mathcal{P}(d\beta)} \right)^2. \quad (3.2)$$

Here we introduce

$$H(\tau) = \mathcal{P}((\beta_d, 1]) + \tau \mathcal{P}(\{\beta_d\}) - \left( \frac{\int_{(\beta_d, 1]} \beta \mathcal{P}(d\beta) + \tau \beta_d \mathcal{P}(\{\beta_d\})}{\int_{I_+} \beta \mathcal{P}(d\beta)} \right)^2.$$

It follows from (1.4) and (1.10) that

$$H(0) \geq 0, \quad H(1) \geq 0. \quad (3.3)$$

Moreover, we have either  $c_d = 0$  or  $c_d = 1$  if  $\mathcal{P}(\{\beta_d\}) > 0$ . In fact, since

$$H''(\tau) = \text{const.} = -2 \left( \frac{\beta_d \mathcal{P}(\{\beta_d\})}{\int_{I_+} \beta \mathcal{P}(d\beta)} \right)^2 < 0$$

by  $\mathcal{P}(\{\beta_d\}) > 0$ , it holds that  $H(\tau) > 0$  for  $0 < \tau < 1$  by (3.3). On the other hand,  $H(c_d) \leq 0$  by (3.2).

We now claim

$$\tilde{\psi} = \psi_d = \chi_{I_d} \quad \mathcal{P}\text{-a.e. on } I_+, \quad (3.4)$$

where

$$I_d = \begin{cases} [\beta_d, 1] & \text{if } \mathcal{P}(\{\beta_d\}) = 0 \text{ or if } \mathcal{P}(\{\beta_d\}) > 0 \text{ and } c_d = 1 \\ (\beta_d, 1] & \text{otherwise (i.e., } \mathcal{P}(\{\beta_d\}) > 0 \text{ and } c_d = 0). \end{cases}$$

First, we assume that  $\mathcal{P}(\{\beta_d\}) = 0$ . Then,  $H(\tau) = H(0)$  for  $\tau \in [0, 1]$ . In this case, the equality holds in (3.2) by (3.3), and thus

$$d = \left( \int_{I_+} \phi_0(\beta) \psi_d(\beta) \mathcal{P}(d\beta) \right)^2 = \left( \int_{I_+} \phi_0(\beta) \tilde{\psi}(\beta) \mathcal{P}(d\beta) \right)^2,$$

which means  $\tilde{\psi} = \psi_d$   $\mathcal{P}$ -a.e. on  $I_+$  by the uniqueness of Lemma 2.9. Note that the integrands are non-negative. It is clear that  $\psi_d = \chi_{I_d}$   $\mathcal{P}$ -a.e. on  $I_+$ . Next we assume that  $\mathcal{P}(\{\beta_d\}) > 0$ . Then, we use (3.2)-(3.3) to obtain  $H(c_d) = 0$ , which again implies that the equality holds in (3.2), and hence

$$\tilde{\psi} = \psi_d = \begin{cases} \chi_{[\beta_d, 1]} & \text{if } c_d = 1 \\ \chi_{(\beta_d, 1]} & \text{if } c_d = 0. \end{cases}$$

The claim (3.4) is established.

Property (3.4) is actually refined as follows, recall (2.20), i.e.,

$$\mathcal{B} = I_{\inf} \cap \text{supp} \mathcal{P}.$$

**Lemma 3.1.**  $\tilde{\psi} = \chi_{I_{\inf}}$   $\mathcal{P}$ -a.e. on  $I_+$ .

*Proof.* There are the following six possibilities:

- (i)  $\beta_d < \beta_{\inf}$     (ii)  $\beta_d > \beta_{\inf}$
- (iii)  $\beta_d = \beta_{\inf}$ ,  $I_d = (\beta_d, 1]$  and  $\beta_{\inf} \in I_{\inf}$
- (iv)  $\beta_d = \beta_{\inf}$ ,  $I_d = [\beta_d, 1]$  and  $\beta_{\inf} \notin I_{\inf}$
- (v)  $\beta_d = \beta_{\inf}$ ,  $I_d = [\beta_d, 1]$  and  $\beta_{\inf} \in I_{\inf}$
- (vi)  $\beta_d = \beta_{\inf}$ ,  $I_d = (\beta_d, 1]$  and  $\beta_{\inf} \notin I_{\inf}$

The lemma is clearly true for the cases (v)-(vi), and thus it suffices to prove  $\mathcal{P}(I_d \setminus I_{\inf}) = 0$ ,  $\mathcal{P}(I_{\inf} \setminus I_d) = 0$  and  $\mathcal{P}(\{\beta_d\}) = \mathcal{P}(\{\beta_{\inf}\}) = 0$  for the cases (i), (ii) and (iii)-(iv), respectively.

(i) Assume  $\mathcal{P}(I_d \setminus I_{\inf}) > 0$ . Then,

$$\tilde{\psi}(\beta) = 0 \quad \text{for } \beta \in I_d \setminus I_{\inf} \quad (3.5)$$



by the definitions of  $I_{\inf}$  and  $\tilde{\psi}$ . Note that  $\tilde{w}_{k,\beta} \rightarrow -\infty$  locally uniformly in  $\mathbf{R}^2$  for  $\beta \in I_d \setminus I_{\inf}$ . On the other hand,  $\tilde{\psi}(\beta) = 1$  for some  $\beta \in I_d \setminus I_{\inf}$  by (3.4), which contradicts (3.5).

(ii) Assume  $\mathcal{P}(I_{\inf} \setminus I_d) > 0$ . Then,

$$\tilde{\psi}(\beta) = 0 \quad \text{for } \mathcal{P}\text{-a.e. } \beta \in I_{\inf} \setminus I_d \quad (3.6)$$

by (3.4). On the other hand,  $\tilde{\psi}(\beta) > 0$  for any  $\beta \in I_{\inf} \setminus I_d$  by the definitions of  $I_{\inf}$  and  $\tilde{\psi}$ , and by the convergence (2.21), which contradicts (3.6).

(iii) If  $\mathcal{P}(\{\beta_d\}) = \mathcal{P}(\{\beta_{\inf}\}) > 0$  then  $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{\inf}) = 0$  by (3.4) and  $I_d = (\beta_d, 1]$ . On the other hand,  $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{\inf}) > 0$  by  $\beta_{\inf} \in I_{\inf}$  as shown for the case (ii) above, a contradiction.

(iv) If  $\mathcal{P}(\{\beta_d\}) = \mathcal{P}(\{\beta_{\inf}\}) > 0$  then  $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{\inf}) = 1$  by (3.4) and  $I_d = [\beta_d, 1]$ . On the other hand,  $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{\inf}) = 0$  by  $\beta_{\inf} \notin I_{\inf}$  as shown for the case (i) above, a contradiction.  $\square$

Since the equality holds in (3.2) as shown above, it follows from (3.4) and Lemma 3.1 that

$$\mathcal{P}(I_{\inf}) > 0, \quad \frac{\mathcal{P}(I_{\inf})}{\left(\int_{I_{\inf}} \beta \mathcal{P}(d\beta)\right)^2} = \frac{1}{\left(\int_{I_+} \beta \mathcal{P}(d\beta)\right)^2} \quad (3.7)$$

**Lemma 3.2.** *For every  $R > 0$  and  $\alpha \in I_+ \setminus I_{\inf}$ , it holds that*

$$\lim_{k \rightarrow \infty} \int_{B_{\sigma_{k,\alpha} R}} e^{w_{k,\alpha}} = 0,$$

where  $\sigma_{k,\alpha} = e^{-w_{k,\alpha}(0)/2}$ .

*Proof.* Fix  $R > 0$  and  $\alpha \in I_+ \setminus I_{\inf}$ . Putting

$$\tilde{w}_{k,\alpha}^{(1)}(x) = w_{k,\alpha}(\sigma_{k,\alpha} x) + 2 \log \sigma_{k,\alpha},$$

we have

$$\int_{B_{\sigma_{k,\alpha} R}} e^{w_{k,\alpha}} = \int_{B_R} e^{\tilde{w}_{k,\alpha}^{(1)}}, \quad \tilde{w}_{k,\alpha}^{(1)} \leq \tilde{w}_{k,\alpha}^{(1)}(0) = 0 \quad \text{in } B_R,$$

and therefore it suffices to show

$$\tilde{w}_{k,\alpha}^{(1)} \rightarrow -\infty \quad \text{locally uniformly in } B_R \setminus \{0\}.$$

If this is not the case, then there exist  $C_1 > 0$  and  $r_1 > 0$  such that

$$\max_{\overline{B_R} \setminus B_{r_1}} \tilde{w}_{k,\alpha}^{(1)} \geq -C_1$$

for  $k \gg 1$ . Since there exists  $y_k \in \overline{B_R} \setminus B_{r_1}$  such that

$$\tilde{w}_{k,\alpha}^{(1)}(y_k) = \max_{\overline{B_R} \setminus B_{r_1}} \tilde{w}_{k,\alpha}^{(1)},$$

it holds that

$$\tilde{w}_{k,\alpha}^{(1)}(y_k) - \tilde{w}_{k,\alpha}^{(1)}(0) \geq -C_1$$

for  $k \gg 1$ , and thus

$$w_k(\sigma_{k,\alpha} y_k) - w_k(0) = (\tilde{w}_{k,\alpha}^{(1)}(y_k) - \tilde{w}_{k,\alpha}^{(1)}(0))/\alpha \geq -C_1/\alpha \quad (3.8)$$

for  $k \gg 1$ . On the other hand, we have  $\beta \in I_{\inf}$  satisfying

$$\lim_{L \rightarrow +\infty} \lim_{k \rightarrow \infty} \int_{B_{\sigma_k L}} e^{w_{k,\beta}} = 1 \quad (3.9)$$

by the definitions of  $I_{\inf}$  and  $\tilde{\psi}$ , and by the convergence shown in the proof of Lemma 2.5.

Now, we introduce

$$\tilde{w}_k^{(2)}(x) = w_{k,\beta}(\mu_k x + \sigma_{k,\alpha} y_k) + 2 \log \mu_k, \quad \mu_k = e^{-w_{k,\beta}(\sigma_{k,\alpha} y_k)/2}.$$

Since  $\beta \in I_{\inf}$ , there exists  $C_2 > 0$  such that

$$w_k(0) \leq w_{k,\beta}(0) + C_2 \quad (3.10)$$

for  $k \gg 1$ . Moreover, it follows from (3.8) and (3.10) that

$$w_k(0) - w_{k,\beta}(\sigma_{k,\alpha} y_k) \leq C_3 \quad (3.11)$$

for  $k \gg 1$ , where  $C_3 = \beta C_1/\alpha + C_2$ . Noting (3.11) and

$$w_k(0) = \sup_{\alpha \in I_+} \sup_{x \in \Omega} w_{k,\alpha}(x),$$

we find

$$-\Delta \tilde{w}_k^{(2)} = \beta \lambda_k e^{\tilde{\xi}_k^{(2)}} \int_{I_+} \tau \left( e^{\tilde{w}_{k,\tau}^{(2)}} - \frac{\mu_k^2}{|\Omega|} \right) \mathcal{P}(d\tau) \quad \text{in } B_{\frac{r_1}{2\mu_k}} \quad (3.12)$$

$$\tilde{w}_k^{(2)}(0) = 0, \quad \tilde{w}_{k,\tau}^{(2)} \leq w_k(0) - w_{k,\beta}(\sigma_{k,\alpha} y_k) \leq C_3 \quad \text{in } B_{\frac{r_1}{2\mu_k}} \text{ for any } \tau \in I_+$$

$$\int_{B_{\frac{r_1}{2\mu_k}}} e^{\tilde{w}_{k,\tau}^{(2)} + \tilde{\xi}_k^{(2)}} \leq 1 \quad \text{for any } \tau \in I_+,$$

where

$$\tilde{w}_{k,\tau}^{(2)}(x) = w_{k,\tau}(\mu_k x + \sigma_{k,\alpha} y_k) + 2 \log \mu_k, \quad \tilde{\xi}_k^{(2)}(x) = \xi_k(\mu_k x + \sigma_{k,\alpha} y_k).$$

Noting  $\sigma_{k,\alpha} y_k \rightarrow 0$ ,  $\xi_k(0) = 0$  and the smoothness of  $\xi_k$ , we perform the compactness argument, similarly to the proof of Proposition 5, to obtain  $\tilde{w}^{(2)}, \tilde{f}^{(2)} \in C^2(\mathbf{R}^2)$  and  $C_4$  such that

$$\tilde{w}_k^{(2)} \rightarrow \tilde{w}^{(2)}, \quad [\text{r.h.s. of (3.12)}] \rightarrow \tilde{f}^{(2)} \quad \text{in } C_{loc}^2(\mathbf{R}^2)$$

and

$$\begin{aligned} -\Delta \tilde{w}^{(2)} &= \tilde{f}^{(2)}, \quad \tilde{w}^{(2)} \leq C_3 \quad \text{in } \mathbf{R}^2 \\ \tilde{w}^{(2)}(0) &= 0, \quad \int_{\mathbf{R}^2} e^{\tilde{w}^{(2)}} \leq 1, \quad \int_{\mathbf{R}^2} \tilde{f}^{(2)} + \|\tilde{f}^{(2)}\|_{L^\infty(\mathbf{R}^2)} \leq C_4. \end{aligned}$$

Therefore, there exist  $\ell_1 > 0$  and  $0 < \delta \ll 1$  such that

$$\int_{B_{\mu_k \ell_1}(\sigma_{k,\alpha} y_k)} e^{w_{k,\beta}} \geq 2\delta \quad (3.13)$$

for  $k \gg 1$ .

On the other hand, (3.9) admits  $\ell_2 > 0$  satisfying

$$\int_{B_{\sigma_k \ell_2}} e^{w_{k,\beta}} \geq 1 - \delta \quad (3.14)$$

for  $k \gg 1$ . In addition,

$$|\sigma_{k,\alpha} y_k| - \mu_k \ell_1 - \sigma_k \ell_2 \geq \sigma_{k,\alpha} \left( r_1 - \frac{\mu_k}{\sigma_{k,\alpha}} \ell_1 - \frac{\sigma_k}{\sigma_{k,\alpha}} \ell_2 \right) \geq \frac{\sigma_{k,\alpha} r_1}{2}$$

for  $k \gg 1$  since

$$\frac{\sigma_k}{\sigma_{k,\alpha}} \rightarrow 0, \quad \frac{\mu_k}{\sigma_{k,\alpha}} \rightarrow 0$$

by  $\alpha \notin I_{\inf}$  and (3.11). Hence there holds

$$B_{\mu_k \ell_1}(\sigma_{k,\alpha} y_k) \cap B_{\sigma_k \ell_2} = \emptyset \quad (3.15)$$

for  $k \gg 1$ . Combining (3.13)-(3.15) shows

$$1 = \int_{\Omega} e^{w_{k,\beta}} \geq \int_{B_{\mu_k \ell_1}(\sigma_{k,\alpha} y_k) \cup B_{\sigma_k \ell_2}} e^{w_{k,\beta}} \geq 2\delta + (1 - \delta) = 1 + \delta > 1$$

for  $k \gg 1$ , a contradiction.  $\square$

**Lemma 3.3.** *There are no  $\mathcal{P}$ -measurable sets  $K_1, K_2 \subset I_+$  satisfying*

$$\begin{cases} \mathcal{P}(K_i) > 0 \ (i = 1, 2), & \mathcal{P}(K_1 \cap K_2) = 0 \\ \frac{\mathcal{P}(K_i)}{\left(\int_{K_i} \beta \mathcal{P}(d\beta)\right)^2} = \frac{1}{\left(\int_{I_+} \beta \mathcal{P}(d\beta)\right)^2} \ (i = 1, 2). \end{cases} \quad (3.16)$$

*Proof.* Assume that there exist  $\mathcal{P}$ -measurable sets  $K_1, K_2 \subset I_+$  satisfying (3.16), and put

$$a_i = \mathcal{P}(K_i), \quad b_i = \frac{\int_{K_i} \beta \mathcal{P}(d\beta)}{\int_{I_+} \beta \mathcal{P}(d\beta)},$$

so that

$$a_i = b_i^2 \quad (i = 1, 2).$$

On the other hand, (1.4) and (1.10) show

$$\frac{1}{\left(\int_{I_+} \beta \mathcal{P}(d\beta)\right)^2} \leq \frac{\mathcal{P}(K_1 \cup K_2)}{\left(\int_{K_1 \cup K_2} \beta \mathcal{P}(d\beta)\right)^2} = \frac{\mathcal{P}(K_1) + \mathcal{P}(K_2)}{\left(\int_{K_1} \beta \mathcal{P}(d\beta) + \int_{K_2} \beta \mathcal{P}(d\beta)\right)^2}$$

or

$$\frac{a_1 + a_2}{(b_1 + b_2)^2} \geq 1.$$

Hence we have

$$b_1^2 + b_2^2 \geq (b_1 + b_2)^2,$$

which is impossible since  $b_i > 0$  ( $i = 1, 2$ ).  $\square$

**Lemma 3.4.** *There exists  $C_5 > 0$ , independent of  $k \gg 1$ , such that*

$$\sup_{\alpha \in I_+} \sup_{x \in B_{2R_0}} \{w_{k,\alpha}(x) + 2 \log |x|\} \leq C_5 \quad (3.17)$$

for  $k \gg 1$ .

*Proof.* The proof is divided into four steps.

*Step 1.* Assume the contrary, that is, there exist  $\alpha_k \in I_+$  and  $x_k \in \bar{B}_{R_0}$  such that

$$M_k \equiv w_{k,\alpha_k}(x_k) + 2 \log |x_k| = \sup_{\alpha \in I_+} \sup_{x \in B_{2R_0}} \{w_{k,\alpha}(x) + 2 \log |x|\} \rightarrow +\infty.$$

We have

$$\begin{aligned} w_{k,\alpha_k}(x_k) &= M_k - 2 \log |x_k| \geq M_k - 2 \log R_0 \rightarrow +\infty \\ \ell_k &\equiv e^{w_{k,\alpha_k}(x_k)/2} \cdot \frac{|x_k|}{2} = \frac{e^{M_k/2}}{2} \rightarrow +\infty. \end{aligned}$$

For any  $x \in B_{|x_k|/2}(x_k)$ ,  $\alpha \in I_+$  and  $k$ , it holds that

$$\begin{aligned} w_{k,\alpha}(x) - w_{k,\alpha_k}(x_k) \\ = (w_{k,\alpha}(x) + 2 \log |x|) - (w_{k,\alpha_k}(x_k) + 2 \log |x_k|) + 2 \log \frac{|x_k|}{|x|} \leq 2 \log 2. \end{aligned}$$

We put

$$\hat{w}_k(x) = w_{k,\alpha_k}(\tau_k x + x_k) + 2 \log \tau_k, \quad \tau_k = e^{-w_{k,\alpha_k}(x_k)/2},$$

and get

$$\begin{aligned} -\Delta \hat{w}_k &= \hat{f}_k - \hat{\delta}_k e^{\hat{\xi}_k} \quad \text{in } B_{\ell_k} \\ \hat{w}_{k,\beta} &\leq 2 \log 2 \quad \text{in } B_{\ell_k} \text{ for any } \beta \in I_+ \text{ and } k \\ \hat{w}_k(0) &= 0, \quad \int_{B_{\ell_k}} e^{\hat{w}_{k,\beta} + \hat{\xi}_k} \leq 1 \quad \text{for any } \beta \in I_+ \text{ and } k, \end{aligned} \quad (3.18)$$

where

$$\begin{aligned} \hat{f}_k(x) &= \alpha_k \lambda_k \int_{I_+} \beta e^{\hat{w}_{k,\beta} + \hat{\xi}_k} \mathcal{P}(d\beta), \quad \hat{\delta}_k = \frac{\alpha_k \lambda_k \tau_k^2 \int_{I_+} \beta \mathcal{P}(d\beta)}{|\Omega|}, \\ \hat{w}_{k,\beta}(x) &= w_{k,\beta}(\tau_k x + x_k) + 2 \log \tau_k, \quad \hat{\xi}_k(x) = \xi_k(\tau_k x + x_k). \end{aligned}$$

The compactness argument, similarly to the proof of Proposition 5, admits  $\hat{w}, \hat{f} \in C^2(\mathbf{R}^2)$  and  $C_6 > 0$  such that

$$\hat{w}_k \rightarrow \hat{w}, \quad \hat{f}_k \rightarrow \hat{f} \quad \text{in } C_{loc}^2(\mathbf{R}^2) \quad (3.19)$$

and

$$\begin{aligned} -\Delta \hat{w} &= \hat{f} \neq 0, \quad \hat{w} \leq 2 \log 2 \quad \text{in } \mathbf{R}^2, \\ \hat{w}(0) &= 0, \quad \int_{\mathbf{R}^2} e^{\hat{w}} \leq 1, \quad \|\hat{f}\|_{L^\infty(\mathbf{R}^2)} + \int_{\mathbf{R}^2} \hat{f} \leq C_6. \end{aligned} \quad (3.20)$$

Note that

$$\alpha_0 = \lim_{k \rightarrow \infty} \alpha_k \neq 0 \quad (3.21)$$

by the Liouville theorem. Since  $\hat{f} \in L^1 \cap L^\infty(\mathbf{R}^2)$ , the function

$$\hat{z}(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \hat{f}(y) \log \frac{|x-y|}{1+|y|} dy \quad (3.22)$$

is well-defined, and satisfies

$$\frac{\hat{z}(x)}{\log|x|} \rightarrow \hat{\gamma} = \frac{1}{2\pi} \int_{\mathbf{R}^2} \hat{f} \quad \text{as } |x| \rightarrow +\infty \quad (3.23)$$

by Lemma 2.3. Similarly to the proof of Proposition 5, we see

$$\begin{aligned} \hat{w}(x) &= -\hat{z}(x) + \hat{z}(0), \\ \hat{w}(x) &\geq -\hat{\gamma} \log(1+|x|) + \frac{1}{2\pi} \int_{\mathbf{R}^2} \hat{f}(y) \log \frac{|y|}{1+|y|} dy, \end{aligned} \quad (3.24)$$

for any  $x \in \mathbf{R}^2$ , where

$$\hat{\gamma} = \frac{1}{2\pi} \int_{\mathbf{R}^2} \hat{f}.$$

*Step 2.* We introduce

$$\hat{\mathcal{B}} = \{\beta \in \text{supp } \mathcal{P} \mid \limsup_{k \rightarrow \infty} (w_{k,\beta}(x_k) - w_{k,\alpha_k}(x_k)) > -\infty\} \quad (3.25)$$

and put

$$\hat{\beta}_{\inf} = \inf_{\beta \in \hat{\mathcal{B}}} \beta. \quad (3.26)$$

Note that  $\mathcal{P}(\hat{\mathcal{B}}) > 0$ , and so  $\hat{\beta}_{\inf}$  is well-defined, since  $\hat{f} \not\equiv 0$  as in (3.20).

In this step, we shall show

$$\frac{\hat{\beta}_{\inf} \hat{\gamma}}{\alpha_0} > 2, \quad (3.27)$$

where  $\alpha_0$  is as in (3.21).

By the definition of  $\hat{\mathcal{B}}$ , for every  $\beta \in \hat{\mathcal{B}}$ , there exists a subsequence such that  $\hat{w}_{k,\beta}(0) = w_{k,\beta}(x_k) - w_{k,\alpha_k}(x_k) = O(1)$ . It follows from (3.18) that

$$-\Delta \hat{w}_{k,\beta} = \frac{\beta}{\alpha_k} (\hat{f}_k - \hat{\delta}_k e^{\hat{\varepsilon}_k}) \quad \text{in } B_{\ell_k}.$$

We repeat the procedure developed in Step 1 to obtain  $\hat{w}_\beta = \hat{w}_\beta(x) \in C^2(\mathbf{R}^2)$  satisfying

$$\begin{aligned} \hat{w}_{k,\beta} &\rightarrow \hat{w}_\beta \quad \text{in } C_{loc}^2(\mathbf{R}^2), \\ -\Delta \hat{w}_\beta &= \frac{\beta}{\alpha_0} \hat{f}, \quad \hat{w}_\beta \leq \hat{w}_\beta(0) \leq 0 \quad \text{in } \mathbf{R}^2, \quad \int_{\mathbf{R}^2} e^{\hat{w}_\beta} \leq 1, \end{aligned}$$

and

$$\hat{w}_\beta(x) \geq -\frac{\beta}{\alpha_0} \hat{\gamma} \log(1+|x|) + \frac{\beta}{2\pi\alpha_0} \int_{\mathbf{R}^2} \hat{f}(y) \log \frac{|y|}{1+|y|} dy, \quad (3.28)$$

where  $\hat{f} = \hat{f}(x)$  is the limit function in (3.19).

If  $\hat{\beta}_{\inf} \in \hat{\mathcal{B}}$  then we take  $\beta = \hat{\beta}_{\inf}$ , and obtain (3.27) by (3.28) and

$$\hat{f} \in L^1 \cap L^\infty(\mathbf{R}^2), \quad \int_{\mathbf{R}^2} e^{\hat{w}_\beta} \leq 1.$$

If  $\hat{\beta}_{\inf} \notin \hat{\mathcal{B}}$  then we take  $\beta_j \in \hat{\mathcal{B}}$  satisfying  $\beta_j \downarrow \hat{\beta}_{\inf}$ , and obtain  $\varepsilon_1 > 0$  independent of  $j$  such that  $\beta_j \tilde{\gamma} \geq 2 + \varepsilon_1$ , using (3.28) for  $\beta = \beta_j$  and

$$\frac{\beta_j}{2\pi} \int_{\mathbf{R}^2} \hat{f}(y) \log \frac{|y|}{1+|y|} dy = O(1), \quad \int_{\mathbf{R}^2} e^{\hat{w}_{\beta_j}} \leq 1,$$

and thus (3.27) is shown.

*Step 3.* Given a bounded open set  $\omega \subset \mathbf{R}^2$ , it holds that

$$\int_{I_+} \left( \int_{\omega} e^{\hat{w}_{k,\beta} + \hat{\xi}_k} dx \right) \mathcal{P}(d\beta) \leq 1,$$

and hence

$$\left( \int_{\omega} e^{\hat{w}_{k,\beta} + \hat{\xi}_k} dx \right) \mathcal{P}(d\beta) \xrightarrow{*} \hat{\zeta}^\omega(d\beta) \quad \text{in } \mathcal{M}(I_+)$$

for some  $\hat{\zeta}^\omega \in \mathcal{M}(I_+)$ . Similarly to the proof of Lemma 2.6, we see that there exists  $\hat{\psi}^\omega \in L^1(I_+, \mathcal{P})$  such that  $0 \leq \hat{\psi}^\omega \leq 1$   $\mathcal{P}$ -a.e. on  $I_+$  and

$$\hat{\zeta}^\omega(\eta) = \int_{\eta} \hat{\psi}^\omega(\beta) \mathcal{P}(d\beta)$$

for any Borel set  $\eta \subset I_+$ . We take  $R_j \uparrow +\infty$  and put  $\omega_j = B_{R_j}$ . From the monotonicity of  $\hat{\psi}^\omega$  with respect to  $\omega$ , there exist  $\hat{\zeta} \in \mathcal{M}(I_+)$  and  $\hat{\psi} \in L^1(I_+, \mathcal{P})$  such that

$$\begin{aligned} 0 &\leq \hat{\psi}(\beta) \leq 1, \quad \mathcal{P}\text{-a.e. } \beta \\ 0 &\leq \hat{\psi}^{\omega_1}(\beta) \leq \hat{\psi}^{\omega_2}(\beta) \leq \dots \rightarrow \hat{\psi}(\beta), \quad \mathcal{P}\text{-a.e. } \beta \\ \hat{\zeta}(\eta) &= \int_{\eta} \hat{\psi}(\beta) \mathcal{P}(d\beta) \quad \text{for any Borel set } \eta \subset I_+. \end{aligned}$$

It follows from (3.19) that

$$\alpha_0 \bar{\lambda} \int_{I_+} \beta \hat{\psi}^{\omega_j}(\beta) \mathcal{P}(d\beta) = \lim_{k \rightarrow \infty} \alpha_k \lambda_k \int_{I_+} \beta \left( \int_{\omega_j} e^{\hat{w}_{k,\beta} + \hat{\xi}_k} dx \right) \mathcal{P}(d\beta) = \int_{\omega_j} \hat{f},$$

and thus we obtain

$$\hat{\gamma} = \frac{1}{2\pi} \int_{\mathbf{R}^2} \hat{f} = \frac{\alpha_0 \bar{\lambda}}{2\pi} \int_{I_+} \beta \hat{\psi}(\beta) \mathcal{P}(d\beta), \quad (3.29)$$

sending  $j \rightarrow \infty$ .

Complying the proof of Lemma 2.7, one can show that

$$\lim_{r \rightarrow +\infty} r \hat{w}_r = -\hat{\gamma}, \quad \lim_{r \rightarrow +\infty} \hat{w}_\theta = 0, \quad (3.30)$$

by using (3.22), (3.24), (3.19), (3.25)-(3.26), (3.23), (3.27) and the property, derived from (3.19), (3.24) and (3.21), that

$$\begin{aligned}\hat{w}_{k,\beta}(x) &= \frac{\beta}{\alpha_k} \hat{w}_k(x) + (w_{k,\beta}(x_k) - w_{k,\alpha_k}(x_k)) \\ &\leq \frac{\beta}{\alpha_k} \hat{w}_k(x) = \frac{\beta}{\alpha_0} (\hat{z}(x) - \hat{z}(0)) + o(1)\end{aligned}$$

as  $k \rightarrow \infty$ , locally uniformly in  $x \in \mathbf{R}^2$ , for any  $\beta \in I_+$ , where  $(r, \theta)$  denotes the polar coordinate in  $\mathbf{R}^2$ . Then, following the proof of Lemma 2.8, we use the Pohozaev identity (2.25), (3.30), (3.29) and the value  $\bar{\lambda}$  given in (1.10) to obtain

$$\int_{I_+} \hat{\psi}(\beta) \mathcal{P}(d\beta) = \left( \int_{I_+} \hat{\phi}_0(\beta) \hat{\psi}(\beta) \mathcal{P}(d\beta) \right)^2, \quad (3.31)$$

where

$$\hat{\phi}_0(\beta) = \frac{\sqrt{\alpha_0}}{\int_{I_+} \alpha \mathcal{P}(d\alpha)} \beta.$$

*Step 4.* In this final step, we shall show that there exists a  $\mathcal{P}$ -measurable set  $J \subset I_+$  such that  $\hat{\psi} = \chi_J$   $\mathcal{P}$ -a.e. on  $I_+$  and that

$$\mathcal{P}(J) > 0, \quad \mathcal{P}(J \cap I_{\inf}) = 0, \quad \frac{\mathcal{P}(J)}{(\int_J \beta \mathcal{P}(d\beta))^2} = \frac{1}{\left(\int_{I_+} \beta \mathcal{P}(d\beta)\right)^2}. \quad (3.32)$$

The proof of the lemma is reduced to showing (3.32) since (3.7) and (3.32) do not occur simultaneously by Lemma 3.3.

Noting that

$$\tilde{\psi} = \chi_{I_{\inf}} \quad \mathcal{P}\text{-a.e. on } I_+,$$

recall Lemma 3.1, and that

$$B_{\tau_k R}(x_k) \cap B_{\sigma_k R} = \emptyset, \quad \int_{\Omega} e^{w_{k,\beta}} = 1$$

for any  $k \gg 1$ ,  $\beta \in I_+ \setminus \{0\}$  and  $R > 0$ , we find

$$0 \leq \tilde{\psi} + \hat{\psi} \leq 1 \quad \mathcal{P}\text{-a.e. on } I_+,$$

and thus

$$\hat{\psi} = 0 \quad \mathcal{P}\text{-a.e. on } I_{\inf}. \quad (3.33)$$

We put

$$\hat{I} = I_+ \setminus I_{\inf}$$

and see from (3.31) and (3.33) that

$$\hat{d} = \int_{\hat{I}} \hat{\psi}(\beta) \mathcal{P}(d\beta) = \left( \int_{\hat{I}} \hat{\phi}_0(\beta) \hat{\psi}(\beta) \mathcal{P}(d\beta) \right)^2 > 0. \quad (3.34)$$

Let

$$\begin{aligned}\hat{\mathcal{I}}(\psi) &= \int_{\hat{I}} \hat{\phi}_0(\beta) \psi(\beta) \mathcal{P}(d\beta) \\ \hat{\mathcal{C}} &= \{\psi \mid 0 \leq \psi(\beta) \leq 1 \text{ } \mathcal{P}\text{-a.e. on } \hat{I} \text{ and } \int_{\hat{I}} \psi(\beta) \mathcal{P}(d\beta) = \hat{d}\}.\end{aligned}$$

Noting the monotonicity of  $\hat{\phi}_0 = \hat{\phi}_0(\beta)$  and complying the proof of Lemma 2.9, we can show the following properties:

(a) The value  $\sup_{\psi \in \hat{\mathcal{C}}} \hat{\mathcal{I}}(\psi)$  is attained by

$$\psi_*(\beta) = \chi_{\{\hat{\phi}_0 > \hat{s}\} \cap \hat{I}}(\beta) + \hat{c} \chi_{\{\hat{\phi}_0 = \hat{s}\}}(\beta)$$

with  $\hat{s}$  and  $\hat{c}$  defined by

$$\begin{aligned} \hat{s} &= \inf\{t \mid \mathcal{P}(\{\hat{\phi}_0 > t\} \cap \hat{I}) \leq \hat{d}\} \\ \hat{c} \mathcal{P}(\{\hat{\phi}_0 = \hat{s}\}) &= \hat{d} - \mathcal{P}(\{\hat{\phi}_0 > \hat{s}\} \cap \hat{I}), \quad 0 \leq \hat{c} \leq 1. \end{aligned}$$

(b) The uniqueness holds in the sense that if  $\psi_m \in \hat{\mathcal{C}}$  is the maximizer then  $\psi_m = \psi_*$   $\mathcal{P}$ -a.e. on  $\hat{I}$ .

Following the argument to show (3.4), which is developed in the first part of the present section, and using (1.4), (1.10), (3.33) and properties (a)-(b), we find that there exists  $\hat{\beta} \in I_+$  such that

$$\hat{\psi} = \chi_{\hat{J}} \quad \mathcal{P}\text{-a.e. on } \hat{I}, \quad (3.35)$$

where

$$\hat{J} = \begin{cases} [\hat{\beta}, 1] \setminus I_{\inf} & \text{if } \mathcal{P}(\{\hat{\beta}\}) = 0 \text{ or if } \mathcal{P}(\{\hat{\beta}\}) > 0 \text{ and } \hat{c} = 1 \\ (\hat{\beta}, 1] \setminus I_{\inf} & \text{otherwise (i.e., } \mathcal{P}(\{\hat{\beta}\}) > 0 \text{ and } \hat{c} = 0). \end{cases}$$

Note that either  $\hat{c} = 0$  or  $\hat{c} = 1$  holds if  $\mathcal{P}(\{\hat{\beta}\}) > 0$ .

Consequently, (3.34) and (3.35) yield

$$\frac{\mathcal{P}(\hat{J})}{\left(\int_{\hat{J}} \beta \mathcal{P}(d\beta)\right)^2} = \frac{\alpha_0}{\left(\int_{I_+} \alpha \mathcal{P}(d\alpha)\right)^2} \leq \frac{1}{\left(\int_{I_+} \alpha \mathcal{P}(d\alpha)\right)^2},$$

which implies

$$\frac{\mathcal{P}(\hat{J})}{\left(\int_{\hat{J}} \beta \mathcal{P}(d\beta)\right)^2} = \frac{1}{\left(\int_{I_+} \alpha \mathcal{P}(d\alpha)\right)^2}$$

by  $\alpha_0 \leq 1$  and (1.10). Hence (3.32) is shown for  $J = \hat{J}$ .  $\square$

**Lemma 3.5.** *There exist  $t \in (0, 1)$  and  $C_7 > 0$  such that*

$$\sup_{\partial B_r} w_{k,\alpha} \leq C_7 + t \inf_{\partial B_r} w_{k,\alpha} - 2(1-t) \log r$$

for any  $r \in [2r', R_0]$ ,  $r' \in (0, R_0/2]$ ,  $\alpha \in I_+$  and  $k \gg 1$ , where  $t$  and  $C_7$  are independent of  $r$ ,  $r'$ ,  $R_0$ ,  $\alpha$  and  $k \gg 1$ .

*Proof.* We comply [12]. Fix  $r \in [2r', R_0]$  and  $r' \in (0, R_0/2]$ , and put

$$z_{k,\alpha}(x) = w_{k,\alpha}(rx) + 2 \log r$$

for  $\alpha \in I_+$  and  $k$ . Then it holds that

$$-\Delta z_{k,\alpha} = \alpha \lambda_k e^{\xi_k(rx)} \int_{I_+} \beta \left( e^{z_{k,\beta}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\beta) \quad \text{in } B_2 \setminus \overline{B_{1/2}}. \quad (3.36)$$



It follows from Lemma 3.4 that

$$z_{k,\alpha}(x) = (w_{k,\alpha}(rx) + 2 \log(r|x|)) - 2 \log|x| \leq C_5 + 2 \log 2 \quad (3.37)$$

for any  $x \in B_2 \setminus \overline{B_{1/2}}$ ,  $\alpha \in I_+$  and  $k \gg 1$ . Thus there exists  $C_8 > 0$ , independent of  $r, r', R_0, \alpha$  and  $k \gg 1$ , such that

$$\begin{aligned} |[r.h.s. \text{ of (3.36)}]| &\leq \lambda_k \sup_{B_{2R_0}} e^{\xi_k} \left( \frac{1}{|\Omega|} + \sup_{\beta \in I_+} \sup_{x \in B_2 \setminus \overline{B_{1/2}}} e^{z_{k,\beta}} \right) \\ &\leq \lambda_k \sup_{B_{2R_0}} e^{\xi_k} \left( \frac{1}{|\Omega|} + 4e^{C_5} \right) \leq C_8 \quad \text{in } B_2 \setminus \overline{B_{1/2}} \end{aligned} \quad (3.38)$$

for  $\alpha \in I_+$  and  $k \gg 1$  by (3.37).

Let  $z'_{k,\alpha} = z'_{k,\alpha}(x)$  be the unique solution to

$$\begin{aligned} -\Delta z'_{k,\alpha} &= \alpha \lambda_k e^{\xi_k(rx)} \int_{I_+} \beta \left( e^{z_{k,\beta}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\beta) \quad \text{in } B_2 \setminus \overline{B_{1/2}}, \\ z'_{k,\alpha} &= 0 \quad \text{on } \partial(B_2 \setminus \overline{B_{1/2}}). \end{aligned}$$

The elliptic regularity and (3.38) admit  $C_9 > 0$ , independent of  $r, r', R_0, \alpha$  and  $k \gg 1$ , such that

$$|z'_{k,\alpha}| \leq C_9 \quad \text{in } B_2 \setminus \overline{B_{1/2}} \quad (3.39)$$

for  $\alpha \in I_+$  and  $k \gg 1$ . Here we introduce

$$h_{k,\alpha}(x) = C_{10} + (z'_{k,\alpha}(x) - z_{k,\alpha}(x)), \quad C_{10} = C_5 + 2 \log 2 + C_9$$

in view of (3.37) and (3.39). The maximum principle assures that  $h_{k,\alpha} = h_{k,\alpha}(x)$  is the non-negative harmonic function on  $B_2 \setminus \overline{B_{1/2}}$ , and then the Harnack inequality admits a universal constant  $t \in (0, 1)$  such that

$$t \sup_{\partial B_1} h_{k,\alpha} \leq \inf_{\partial B_1} h_{k,\alpha}$$

or

$$t \sup_{\partial B_1} (z'_{k,\alpha} - z_{k,\alpha}) \leq (1-t)C_{10} + \inf_{\partial B_1} (z'_{k,\alpha} - z_{k,\alpha}) \quad (3.40)$$

for  $\alpha \in I_+$  and  $k \gg 1$ . Combining (3.39) and (3.40) shows

$$-tC_9 - t \inf_{\partial B_1} z_{k,\alpha} \leq (1-t)C_{10} + C_9 - \sup_{\partial B_1} z_{k,\alpha},$$

which means the lemma for  $C_7 = (1+t)C_9 + (1-t)C_{10}$ .  $\square$

**Lemma 3.6.** *There exist  $\varepsilon_* > 0$ ,  $R_* > 0$  and  $C_i > 0$  ( $i = 11, 12$ ) such that*

$$w_{k,\alpha}(0) + C_{11} \inf_{\partial B_r} w_{k,\alpha} + 2(1 + C_{11}) \log r \leq C_{12} \quad (3.41)$$

for any  $r \in (0, R_*]$ ,  $\alpha \in [\beta_{\inf} - \varepsilon_*, 1]$  and  $k \gg 1$ , where  $\varepsilon_*$ ,  $R_*$ ,  $C_{11}$  and  $C_{12}$  are independent of  $r, \alpha$  and  $k \gg 1$ .

*Proof.* At first, we note that there exists  $\delta = \delta(\mathcal{P}, I_{\inf}) > 0$  such that

$$\beta_{\inf} = (1 + \delta) \frac{\int_{I_{\inf}} \beta \mathcal{P}(d\beta)}{2\mathcal{P}(I_{\inf})}$$

since  $\beta_{\inf} > \int_{I_{\inf}} \beta \mathcal{P}(d\beta) / (2\mathcal{P}(I_{\inf}))$  by Lemma 2.5, (2.24), Lemma 3.1 and (3.7). We put

$$D = \frac{2}{\delta}$$

and introduce the auxiliary function

$$P_{k,\alpha}(r) = w_{k,\alpha}(0) + \frac{D}{2\pi r} \int_{\partial B_r} w_{k,\alpha} ds + 2(1 + D) \log r$$

inspired by [3, 22]. Since

$$\frac{d}{dr} \left( \frac{1}{2\pi r} \int_{\partial B_r} w_{k,\alpha} ds \right) = \frac{1}{2\pi r} \int_{\partial B_r} \frac{\partial w_{k,\alpha}}{\partial \nu} ds,$$

it holds that

$$\frac{dP_{k,\alpha}}{dr}(r) \leq \frac{D\lambda_k}{2\pi r} Q_{k,\alpha}(r), \quad (3.42)$$

for  $r \in (0, R_0]$  and  $\alpha \in I_+$ , where  $\nu$  is the outer unit normal vector and

$$\begin{aligned} Q_{k,\alpha}(r) &= \frac{4\pi(1 + D)}{D\lambda_k} + \frac{1}{|\Omega|} \int_{B_r} e^{\xi_k} dx \cdot \int_{I_+} \beta \mathcal{P}(d\beta) \\ &\quad - \alpha \int_{I_{\inf}} \beta \left( \int_{B_r} e^{w_{k,\beta} + \xi_k} dx \right) \mathcal{P}(d\beta). \end{aligned}$$

Given  $\varepsilon > 0$  whose range is determined later on, there exists  $R_\varepsilon = R_\varepsilon(\mathcal{P}, \Omega) > 0$  such that

$$\frac{1}{|\Omega|} \int_{B_{R_\varepsilon}} e^{\xi_k} dx \cdot \int_{I_+} \beta \mathcal{P}(d\beta) \leq \varepsilon \quad (3.43)$$

for any  $k$ . We may assume that  $R_\varepsilon$  is monotone increasing in  $\varepsilon$ . We also have  $L_\varepsilon > 0$ , independent of  $r$  and  $k$ , such that

$$\int_{I_{\inf}} \beta \left( \int_{B_r} e^{w_{k,\beta} + \xi_k} dx \right) \mathcal{P}(d\beta) \geq \int_{I_{\inf}} \beta \mathcal{P}(d\beta) - \varepsilon \quad (3.44)$$

for any  $r \geq \sigma_k L_\varepsilon$  and  $k \gg 1$  by the definition of  $\tilde{\psi}$ , Lemma 3.1 and the convergence (2.21). We may assume that  $L_\varepsilon$  is monotone decreasing in  $\varepsilon$ . It is clear that

$$\frac{4\pi(1 + D)}{D\lambda_k} \leq \frac{4\pi(1 + D)}{D\bar{\lambda}} + \varepsilon \quad (3.45)$$

for  $k \gg 1$ . Properties (3.43)-(3.45) imply

$$Q_{k,\alpha}(r) \leq 2\varepsilon + \frac{4\pi(1 + D)}{D\bar{\lambda}} - (\beta_{\inf} - \varepsilon) \left( \int_{I_{\inf}} \beta \mathcal{P}(d\beta) - \varepsilon \right) \quad (3.46)$$

for any  $r \in [\sigma_k L_\varepsilon, R_\varepsilon]$ ,  $\alpha \in [\beta_{\inf} - \varepsilon, 1]$  and  $k \gg 1$ .

We now examine the range of  $\varepsilon$  such that the right-hand-side of (3.46) is non-positive. It follows from (1.10) and (3.7) that

$$\frac{4\pi(1+D)}{D\bar{\lambda}} = (1+1/D) \frac{\left(\int_{I_{\inf}} \beta \mathcal{P}(d\beta)\right)^2}{2\mathcal{P}(I_{\inf})}. \quad (3.47)$$

We use (3.7), (3.47) and  $D = 2/\delta$  to obtain

$$\begin{aligned} & [\text{r.h.s. of (3.46)}] \\ &= -\varepsilon^2 + \left\{ 2 + \left( 1 + \frac{1+\delta}{2\mathcal{P}(I_{\inf})} \right) \int_{I_{\inf}} \beta \mathcal{P}(d\beta) \right\} \varepsilon - \frac{\delta \left( \int_{I_{\inf}} \beta \mathcal{P}(d\beta) \right)^2}{4\mathcal{P}(I_{\inf})}, \end{aligned}$$

and therefore, there exists  $\varepsilon_* = \varepsilon_*(\mathcal{P}, I_{\inf}) > 0$  such that

$$[\text{r.h.s. of (3.46)}] \leq 0 \quad (3.48)$$

for any  $0 < \varepsilon < \varepsilon_*$ .

Noting that  $Q_{k,\alpha}(r)$  is independent of  $\varepsilon$ , we organize (3.42), (3.46) and (3.48), so that  $P'_{k,\alpha}(r) \leq 0$  for any  $r \in [\sigma_k L_{\varepsilon_*}, R_{\varepsilon_*}]$ ,  $\alpha \in [\beta_{\inf} - \varepsilon_*, 1]$  and  $k \gg 1$ . This implies

$$\sup_{0 < r \leq R_*} P_{k,\alpha}(r) = \sup_{0 < r \leq \sigma_k L_*} P_{k,\alpha}(r) \quad (3.49)$$

for  $\alpha \in [\beta_{\inf} - \varepsilon_*, 1]$  and  $k \gg 1$ , where  $R_* = R_{\varepsilon_*}$  and  $L_* = L_{\varepsilon_*}$ . Using  $w_{k,\alpha} \leq w_{k,\alpha}(0) \leq w_k(0)$  valid for any  $\alpha \in I_+$ , we estimate  $P_{k,\alpha}$  by

$$\begin{aligned} P_{k,\alpha}(r) &= (1+D)w_{k,\alpha}(0) + \frac{D}{2\pi r} \int_{\partial B_r} (w_{k,\alpha} - w_{k,\alpha}(0)) ds + 2(1+D) \log r \\ &\leq (1+D)(w_{k,\alpha}(0) + 2 \log \sigma_k L_*) \leq 2(1+D) \log L_* \end{aligned} \quad (3.50)$$

for  $r \in (0, \sigma_k L_*]$ ,  $\alpha \in [\beta_{\inf} - \varepsilon_*, 1]$  and  $k \gg 1$ .

Finally, we obtain  $C_{11} = D = 2/\delta$  and  $C_{12} = 2(1+D) \log L_* = 2(1+2/\delta) \log L_*$  by (3.49), (3.50) and [l.h.s. of (3.41)]  $\leq P_{k,\alpha}(r)$ , provided that  $\varepsilon_*$  and  $R_*$  are given above.  $\square$

We are now in a position to prove Proposition 4.

Proof of Proposition 4: Fix  $\alpha_0 \in I_+$  such that

$$\begin{cases} \alpha_0 \in [\max\{\alpha_{\min}, \beta_{\inf} - \varepsilon_*\}, \beta_{\inf}) & \text{if } \beta_{\inf} > \alpha_{\min} \\ \alpha_0 = \alpha_{\min} > \beta_{\inf} - \varepsilon_* & \text{if } \beta_{\inf} = \alpha_{\min}, \end{cases}$$

recall that  $\beta_{\inf}$ ,  $\varepsilon_*$  and  $\alpha_{\min}$  are as in (2.19), Lemma 3.6 and (1.9), respectively. Note that  $\mathcal{P}(I_+ \setminus I_{\inf}) > 0$  and that  $I_{\inf} = (\beta_{\inf}, 1]$  and  $\mathcal{P}(\{\alpha_{\min}\}) > 0$  if  $\beta_{\inf} = \alpha_{\min}$ . It follows from Lemma 3.2 and the uniform boundedness of  $\xi_k$  that

$$\lim_{k \rightarrow \infty} \int_{B_{\sigma_k, \alpha_0} R} e^{w_{k,\alpha_0} + \xi_k} = 0 \quad (3.51)$$

for any  $R > 0$ , where  $\sigma_{k,\alpha_0} = e^{-w_{k,\alpha_0}(0)/2}$ . In addition, the residual vanishing, the uniform boundedness of  $\xi_k$  and the monotonicity (2.2) imply

$$\lim_{k \rightarrow \infty} \int_{B_{2R_0} \setminus B_{R_*}} e^{w_{k,\alpha_0} + \xi_k} = 0, \quad \lim_{k \rightarrow \infty} \int_{\Omega \setminus \Psi_k^{-1}(B_{2R_0})} e^{w_{k,\alpha_0}} = 0, \quad (3.52)$$

where  $R_*$  is as in Lemma 3.6.

Next, we shall prove

$$\lim_{k \rightarrow \infty} \int_{B_{R_*} \setminus B_{\sigma_{k,\alpha_0}}} e^{w_{k,\alpha_0} + \xi_k} = 0. \quad (3.53)$$

For any  $r = |x| \in [\sigma_{k,\alpha_0}, R_*]$ , we calculate

$$\begin{aligned} w_{k,\alpha_0}(x) &\leq \sup_{\partial B_r} w_{k,\alpha_0} \leq C_7 + t \inf_{\partial B_r} w_{k,\alpha_0} - 2(1-t) \log r \\ &\leq C_7 + \frac{t}{C_{11}} \{-w_{k,\alpha_0}(0) - 2(1+C_{11}) \log r + C_{12}\} - 2(1-t) \log r \\ &= -s w_{k,\alpha_0}(0) - 2(1+s) \log r + C_{13}, \end{aligned}$$

using Lemmas 3.5-3.6, where

$$s = t/C_{11}, \quad C_{13} = C_7 + t C_{12}/C_{11}.$$

Hence it holds that

$$\begin{aligned} \int_{B_{R_*} \setminus B_{\sigma_{k,\alpha_0}}} e^{w_{k,\alpha_0} + \xi_k} &\leq \sup_{B_{R_0}} e^{\xi_k} \cdot e^{C_{13} - s w_{k,\alpha_0}(0)} \int_{B_{R_*} \setminus B_{\sigma_{k,\alpha_0}}} |x|^{-2(1+s)} dx \\ &\leq C_{14} e^{-2s w_{k,\alpha_0}(0)} \int_1^\infty r^{-(1+2s)} dr \rightarrow 0 \end{aligned}$$

as  $k \rightarrow \infty$ , where

$$C_{14} = 2\pi e^{C_{13}} \sup_k \sup_{B_{R_0}} e^{\xi_k}.$$

Consequently, (3.51)-(3.53) yield

$$\lim_{k \rightarrow \infty} \int_{\Omega} e^{w_{k,\alpha_0}} = 0,$$

which is impossible since  $\int_{\Omega} e^{w_{k,\alpha_0}} = 1$  for any  $k$ . The proof is complete.  $\square$

We conclude this section with the following proposition.

**Proposition 6.** *Under the assumptions of Theorem 1 it holds that*

$$\alpha_{\min} > \frac{1}{2} \int_{I_+} \beta \mathcal{P}(d\beta) = \frac{2}{\tilde{\gamma}}. \quad (3.54)$$

*Proof.* It suffices to show that  $\beta_{\inf} = \alpha_{\min}$ . Indeed, if this is the case, (3.54) follows from Lemma 2.5 and (1.16). Since  $\beta_{\inf} \geq \alpha_{\min}$  is obvious, we assume the contrary,  $\beta_{\inf} > \alpha_{\min}$ . Then it holds that  $\text{supp } \tilde{\psi} \subset [\beta_{\inf}, 1]$  by the definitions of  $\beta_{\inf}$  and  $\tilde{\psi}$ , and thus we obtain  $\mathcal{P}([\alpha_{\min}, (\beta_{\inf} + \alpha_{\min})/2]) > 0$  and  $\tilde{\psi} = 0$   $\mathcal{P}$ -a.e. on  $[\alpha_{\min}, (\beta_{\inf} + \alpha_{\min})/2]$ . However, this is impossible by (1.17).  $\square$

## 4 Proof of Proposition 3

Henceforth, we put

$$\bar{w} = \frac{1}{|\Omega|} \int_{\Omega} w.$$

Let  $G = G(x, y)$  be the Green function:

$$-\Delta_x G(\cdot, y) = \delta_y - \frac{1}{|\Omega|} \quad \text{in } \Omega, \quad \int_{\Omega} G(x, y) dx = 0.$$

We begin with the following lemma.

**Lemma 4.1.** *It holds that*

$$w_{k,\alpha} - \bar{w}_{k,\alpha} \rightarrow \alpha \bar{\lambda} \left( \int_{I_+} \beta \mathcal{P}(d\beta) \right) G(\cdot, x_0) \quad \text{in } C_{loc}^2(\Omega \setminus \{x_0\}). \quad (4.1)$$

For every  $\omega \subset\subset \Omega \setminus \{x_0\}$ , there exists  $C_{1,\omega} > 0$ , independent of  $k$  and  $\alpha$ , such that

$$\operatorname{osc}_{\omega} w_{k,\alpha} \equiv \sup_{\omega} w_{k,\alpha} - \inf_{\omega} w_{k,\alpha} \leq C_{1,\omega}. \quad (4.2)$$

*Proof.* Since

$$w_{k,\alpha}(x) - \bar{w}_{k,\alpha} = \alpha \int_{\Omega} G(x, y) \left\{ \lambda_k \int_{I_+} \beta e^{w_{k,\beta}(y)} \mathcal{P}(d\beta) \right\} dy$$

and

$$\lambda_k \int_{I_+} \beta e^{w_{k,\beta}} \mathcal{P}(d\beta) \xrightarrow{*} \bar{\lambda} \left( \int_{I_+} \beta \mathcal{P}(d\beta) \right) \delta_{x_0}$$

by (1.8) with  $s = 0$  and  $\mathcal{S} = \{x_0\}$ , recalling Proposition 2, we have

$$w_{k,\alpha} - \bar{w}_{k,\alpha} \rightarrow \alpha \bar{\lambda} \left( \int_{I_+} \beta \mathcal{P}(d\beta) \right) G(\cdot, x_0)$$

locally uniformly in  $\Omega \setminus \{x_0\}$ . Then the standard argument of elliptic regularity implies (4.1) and (4.2).  $\square$

We decompose  $w_k$  as  $w_k = w_k^{(1)} + w_k^{(2)} + w_k^{(3)}$ , using the solutions  $w_k^{(1)}$ ,  $w_k^{(2)}$  and  $w_k^{(3)}$  to

$$\begin{aligned} -\Delta w_k^{(1)} &= g_k \quad \text{in } B_{2R_0}, \quad w_k^{(1)} = 0 \quad \text{on } \partial B_{2R_0} \\ -\Delta w_k^{(2)} &= h_k \quad \text{in } B_{2R_0}, \quad w_k^{(2)} = 0 \quad \text{on } \partial B_{2R_0} \\ -\Delta w_k^{(3)} &= 0 \quad \text{in } B_{2R_0}, \quad w_k^{(3)} = w_k \quad \text{on } \partial B_{2R_0}, \end{aligned}$$

where

$$\begin{aligned} g_k &= g_k(x) \equiv \lambda_k \int_{I_+} \beta e^{w_{k,\beta}(x) + \xi_k(x)} \mathcal{P}(d\beta) \\ h_k &= h_k(x) \equiv -\frac{\lambda_k}{|\Omega|} \int_{I_+} \beta \mathcal{P}(d\beta) e^{\xi_k(x)}. \end{aligned}$$

By the elliptic regularity there exists  $C_2 > 0$  independent of  $k$  such that

$$-C_2 \leq w_k^{(2)} \leq 0 \quad \text{in } B_{2R_0}.$$

By the maximum principle and Lemma 4.1, we also have  $C_3 > 0$  independent of  $k$  such that

$$\operatorname{osc}_{\bar{B}_{2R_0}} w_k^{(3)} \leq C_3.$$

Thus it holds that

$$w_k(x) - w_k(0) = w_k^{(1)}(x) - w_k^{(1)}(0) + O(1) \quad (4.3)$$

as  $k \rightarrow \infty$  uniformly in  $x \in B_{2R_0}$ .

Let  $G_0 = G_0(x, y)$  be the another Green function defined by

$$-\Delta_x G_0(\cdot, y) = \delta_y \quad \text{in } B_{2R_0}, \quad G_0(\cdot, y) = 0 \quad \text{on } \partial B_{2R_0}.$$

Then it holds that

$$w_k^{(1)}(x) - w_k^{(1)}(0) = \int_{B_{2R_0}} (G_0(x, y) - G_0(0, y)) g_k(y) dy \quad (4.4)$$

for  $x \in B_{2R_0}$ . We have, more precisely,

$$G_0(x, y) = \begin{cases} \Gamma(|x - y|) - \Gamma(\frac{|y|}{2R_0}|x - \bar{y}|), & y \neq 0, y \neq x \\ \Gamma(|x|) - \Gamma(2R_0) & y = 0, y \neq x, \end{cases}$$

using the fundamental solution and the Kelvin transformation:

$$\Gamma(|x|) = \frac{1}{2\pi} \log \frac{1}{|x|}, \quad \bar{y} = \left( \frac{2R_0}{|y|} \right)^2 y,$$

which implies

$$G_0(x, y) - G_0(0, y) = \frac{1}{2\pi} \log \frac{|y|}{|x - y|} - \frac{1}{2\pi} \log \frac{|\bar{y}|}{|x - \bar{y}|}$$

for  $y \in B_{2R_0}$  satisfying  $y \neq x$  and  $y \neq 0$ .

Since

$$\frac{2}{3} \leq \frac{|\bar{y}|}{|x - \bar{y}|} \leq 2, \quad x \in B_{R_0}, \quad y \in B_{2R_0} \setminus \{0\},$$

and since

$$0 \leq \int_{B_{2R_0}} g_k \leq \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta) \cdot \sup_{B_{R_0}} e^{\xi_k} = O(1),$$

we end up with

$$\begin{aligned} & \int_{B_{2R_0}} (G_0(x, y) - G_0(0, y)) g_k(y) dy \\ &= \frac{1}{2\pi} \int_{B_{2R_0}} g_k(y) \log \frac{|y|}{|x - y|} dy + O(1) \end{aligned} \quad (4.5)$$

as  $k \rightarrow \infty$  uniformly in  $x \in B_{R_0}$ .

Consequently, (4.3)-(4.5) yield

$$w_k(x) - w_k(0) = \frac{1}{2\pi} \int_{B_{2R_0}} g_k(y) \log \frac{|y|}{|x-y|} dy + O(1)$$

as  $k \rightarrow \infty$  uniformly in  $x \in B_{R_0}$ . This means

$$\begin{aligned} \tilde{w}_k(x) &= \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k}} \sigma_k^2 g_k(y) \log \frac{|y|}{|\sigma_k x - y|} dy + O(1) \\ &= \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k}} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} dy + O(1) \end{aligned} \quad (4.6)$$

as  $k \rightarrow \infty$  uniformly in  $x \in B_{R_0/\sigma_k}$ , where  $\tilde{f}_k = \tilde{f}_k(y)$  is as in (2.6).

Let  $\tilde{\gamma}$  be as in (1.16), and put

$$\tilde{\gamma}_k = \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k}} \tilde{f}_k. \quad (4.7)$$

To employ the argument of [14], we prepare the following lemma with which  $\tilde{\gamma}_k$  and  $\tilde{\gamma}$  are connected.

**Lemma 4.2.** *It holds that*

$$\lim_{k \rightarrow \infty} \tilde{\gamma}_k = \tilde{\gamma}. \quad (4.8)$$

*Proof.* From (2.6),  $\int_{\Omega} e^{w_k, \beta} = 1$ ,  $\lambda_k \uparrow \bar{\lambda}$  and (1.10), it follows that

$$\tilde{\gamma}_k = \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k}} \tilde{f}_k \leq \frac{\lambda_k}{2\pi} \int_{I_+} \beta \mathcal{P}(d\beta) \leq \tilde{\gamma} \quad (4.9)$$

for any  $k$ . On the other hand, given  $\varepsilon > 0$ , we have  $L_\varepsilon > 0$  such that

$$\liminf_{k \rightarrow \infty} \tilde{\gamma}_k \geq \liminf_{k \rightarrow \infty} \left( \frac{1}{2\pi} \int_{B_{L_\varepsilon}} \tilde{f}_k \right) \geq \tilde{\gamma} - \varepsilon$$

by (2.7), (2.10) and (1.16).  $\square$

**Lemma 4.3.** *For every  $0 < \varepsilon \ll 1$ , there exist  $\tilde{R}_\varepsilon \geq 2$  and  $C_{4,\varepsilon} > 0$  such that*

$$\tilde{w}_k(x) \leq -(\tilde{\gamma}_k - \varepsilon) \log |x| + C_{4,\varepsilon} \quad (4.10)$$

for  $k \gg 1$  and  $x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_\varepsilon}$ .

*Proof.* By (4.8) and (2.7), given  $0 < \varepsilon \ll 1$ , we can take  $\tilde{R}_\varepsilon \geq 2$  such that

$$\frac{1}{2\pi} \int_{B_{\tilde{R}_\varepsilon/2}} \tilde{f}_k \geq \tilde{\gamma}_k - \varepsilon/3 \quad (4.11)$$

for  $k \gg 1$ . It follows from (4.6) that

$$\tilde{w}_k(x) = K_k^1(x) + K_k^2(x) + K_k^3(x) + O(1), \quad k \rightarrow \infty \quad (4.12)$$

uniformly in  $x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_\varepsilon}$ , where

$$\begin{aligned} K_k^1(x) &= \frac{1}{2\pi} \int_{B_{\tilde{R}_\varepsilon/2}} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} dy \\ K_k^2(x) &= \frac{1}{2\pi} \int_{B_{|x|/2}(x)} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} dy \\ K_k^3(x) &= \frac{1}{2\pi} \int_{B'(x)} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} dy \end{aligned}$$

for  $B'(x) = B_{2R_0/\sigma_k} \setminus (B_{\tilde{R}_\varepsilon/2} \cup B_{|x|/2}(x))$ .

Since

$$\frac{|y|}{|x-y|} \leq 2 \frac{|y|}{|x|} \leq \frac{\tilde{R}_\varepsilon}{|x|}, \quad y \in B_{\tilde{R}_\varepsilon/2}, \quad x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_\varepsilon},$$

there exists  $C_{5,\varepsilon} > 0$  independent of  $k \gg 1$  and  $x$  such that

$$K_k^1(x) \leq \frac{1}{2\pi} (\log \tilde{R}_\varepsilon - \log |x|) \int_{B_{\tilde{R}_\varepsilon/2}} \tilde{f}_k \leq C_{5,\varepsilon} - (\tilde{\gamma}_k - \varepsilon/3) \log |x| \quad (4.13)$$

for  $k \gg 1$  and  $x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_\varepsilon}$  by (4.11). We also have

$$\frac{|y|}{|x-y|} \leq 3, \quad y \in B_{2R_0/\sigma_k} \setminus B_{|x|/2}(x),$$

and hence

$$\begin{aligned} K_k^3(x) &\leq \frac{\log 3}{2\pi} \int_{B'(x)} \tilde{f}_k \leq \frac{\log 3}{2\pi} \|\tilde{f}_k\|_{L^1(B_{2R_0/\sigma_k})} \\ &\leq \frac{\lambda_k \log 3}{2\pi} \int_{I_+} \beta \mathcal{P}(d\beta) \cdot \sup_{B_{R_0}} e^{\xi_k} \end{aligned} \quad (4.14)$$

for  $k \gg 1$  and  $x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_\varepsilon}$ .

Now we take

$$D_1(x) = B_{1/|x|}(x), \quad D_2(x) = B_{|x|/2}(x) \setminus B_{1/|x|}(x).$$

Since

$$|y| < |x| + 1/|x|, \quad y \in D_1(x)$$

and

$$\frac{|y|}{|x-y|} \leq \frac{3}{2} |x|^2, \quad y \in D_2(x), \quad x \in \mathbf{R}^2 \setminus B_{\sqrt{2}},$$



we have

$$\begin{aligned}
K_k^2(x) &= \frac{1}{2\pi} \int_{D_1(x) \cup D_2(x)} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} dy \\
&\leq \frac{1}{2\pi} \int_{D_1(x)} \tilde{f}_k(y) \log \frac{|x|+1/|x|}{|x-y|} dy + \frac{2 \log |x| + \log(3/2)}{2\pi} \int_{D_2(x)} \tilde{f}_k \\
&\leq \frac{\|\tilde{f}_k\|_{L^\infty(D_1(x))}}{2\pi} \int_{D_1(x)} \log \frac{1}{|x-y|} dy + \frac{\log(3|x|/2)}{2\pi} \int_{D_1(x)} \tilde{f}_k \\
&\quad + \frac{2 \log |x| + \log(3/2)}{2\pi} \int_{D_2(x)} \tilde{f}_k \\
&\leq \frac{\|\tilde{f}_k\|_{L^\infty(D_1(x))}}{2\pi} \int_{D_1(x)} \log \frac{1}{|x-y|} dy + \frac{\log(3/2)}{2\pi} \int_{B_{|x|/2}(x)} \tilde{f}_k \\
&\quad + \frac{\log |x|}{\pi} \int_{B_{|x|/2}(x)} \tilde{f}_k \leq C_6 + \frac{2\varepsilon}{3} \log |x|
\end{aligned} \tag{4.15}$$

for some  $C_6 > 0$  independent of  $x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_\varepsilon}$ ,  $k \gg 1$ , and  $\varepsilon$ .

Here, the last inequality of (4.15) follows from (4.11) and (4.8). Properties (4.12)-(4.15) imply (4.10).  $\square$

**Lemma 4.4.** *It holds that*

$$\int_{B_{2R_0/\sigma_k}} \tilde{f}_k(y) |\log |y|| dy = O(1) \quad \text{as } k \rightarrow \infty. \tag{4.16}$$

*Proof.* By (3.54) and (4.8), there exist  $\varepsilon_0 > 0$  and  $\delta_0 > 0$  such that

$$-\alpha_{\min}(\tilde{\gamma}_k - \varepsilon_0/2) \leq -(2 + 3\delta_0) \tag{4.17}$$

for  $k \gg 1$ . Let

$$\tilde{R}_0 = \tilde{R}_{\varepsilon_0/2} \tag{4.18}$$

for  $\tilde{R}_\varepsilon$  as in Lemma 4.3 with  $\varepsilon = \varepsilon_0/2$ . Then, by (2.11)-(2.12), (4.10) and (4.17) we obtain  $C_{7,\varepsilon_0} > 0$  such that

$$\begin{aligned}
\tilde{f}_k(y) &= \lambda_k \int_{I_+} \beta e^{\tilde{w}_{k,\beta}(y) + \tilde{\xi}_k(y)} \mathcal{P}(d\beta) \\
&\leq \lambda_k \int_{I_+} \beta e^{\beta \tilde{w}_k(y) + \tilde{\xi}_k(y)} \mathcal{P}(d\beta) \\
&\leq \lambda_k \int_{I_+} \exp \left[ -\beta \{(\tilde{\gamma}_k - \varepsilon_0/2) \log |y| - C_{4,\varepsilon_0}\} + \sup_{B_{2R_0}} \xi_k \right] \mathcal{P}(d\beta) \\
&\leq C_{7,\varepsilon_0} |y|^{-(2+3\delta_0)}
\end{aligned} \tag{4.19}$$

for  $k \gg 1$  and  $y \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_0}$ .

Therefore, we obtain  $C_{8,\varepsilon_0,\delta_0} > 0$  independent of  $k \gg 1$  such that

$$\begin{aligned}
\int_{B_{2R_0/\sigma_k}} \tilde{f}_k(y) |\log |y|| dy &\leq \|\tilde{f}_k\|_{L^\infty(B_{2R_0/\sigma_k})} \int_{B_{\tilde{R}_0}} |\log |y|| dy \\
&\quad + C_{7,\varepsilon_0} \int_{\mathbf{R}^2 \setminus B_{\tilde{R}_0}} |y|^{-(2+3\delta_0)} \log |y| dy \leq C_{8,\varepsilon_0,\delta_0}
\end{aligned}$$

for  $k \gg 1$ , which means (4.16).  $\square$

**Lemma 4.5.** *There exists  $\delta_0 > 0$  such that*

$$\tilde{w}_k(x) = -\tilde{\gamma}_k \log |x| + O(1) \quad \text{as } k \rightarrow \infty$$

*uniformly in  $x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}$ .*

*Proof.* Let  $\varepsilon_0 > 0$  and  $\delta_0 > 0$  satisfy (4.17) and consider

$$\tilde{\gamma}'_k(x) = \frac{1}{2\pi} \int_{B_{|x|/2}} \tilde{f}_k \quad (4.20)$$

for  $x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}$  and  $k \gg 1$ . Since (4.19) holds, there exists  $C_{9,\varepsilon_0,\delta_0} > 0$  such that

$$\begin{aligned} 0 &\leq \tilde{\gamma}_k - \tilde{\gamma}'_k(x) \leq \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k} \setminus B_{\frac{1}{2}(\log \sigma_k^{-1})^{1/\delta_0}}} \tilde{f}_k \\ &\leq \frac{1}{2\pi} \cdot C_{7,\varepsilon_0} \int_{B_{2R_0/\sigma_k} \setminus B_{\frac{1}{2}(\log \sigma_k^{-1})^{1/\delta_0}}} |y|^{-(2+3\delta_0)} dy \\ &\leq C_{9,\varepsilon_0,\delta_0} (\log \sigma_k^{-1})^{-3} \end{aligned} \quad (4.21)$$

for  $x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}$  and  $k \gg 1$ .

Similarly we have

$$\begin{aligned} &\left| \int_{B_{2R_0/\sigma_k} \setminus B_{|x|/2}} \tilde{f}_k(y) \log \frac{1}{|x-y|} dy \right| \\ &= \int_{(B_{2R_0/\sigma_k} \setminus B_{|x|/2}) \cap \{|y-x| \leq 1\}} \tilde{f}_k(y) \log \frac{1}{|x-y|} dy \\ &\quad + \int_{(B_{2R_0/\sigma_k} \setminus B_{|x|/2}) \cap \{|y-x| > 1\}} \tilde{f}_k(y) \log |x-y| dy \\ &\leq C_{7,\varepsilon_0} \left\{ (|x|-1)^{-(2+3\delta_0)} \int_{B_1} \log \frac{1}{|y|} dy \right. \\ &\quad \left. + \int_{(\mathbf{R}^2 \setminus B_{|x|/2}) \cap \{|y-x| > 1\}} |y|^{-(2+3\delta_0)} \log |x-y| dy \right\} \equiv I. \end{aligned} \quad (4.22)$$

Since

$$|y|^{-\delta_0} \log |x-y| \leq |y|^{-\delta_0} \log(|x| + |y|) \leq |y|^{-\delta_0} \log(3|y|)$$

for

$$y \in (\mathbf{R}^2 \setminus B_{|x|/2}) \cap \{|y-x| > 1\}, \quad x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}} \quad (4.23)$$

and  $k \gg 1$ , we have  $C_{10,\delta_0} > 0$  such that

$$|y|^{-(2+3\delta_0)} \log |x-y| \leq C_{10,\delta_0} |y|^{-2(1+\delta_0)}$$

for  $(x, y)$  in (4.23) with  $k \gg 1$ . Hence we have  $C_{11,\varepsilon_0,\delta_0} > 0$  such that

$$I \leq C_{11,\varepsilon_0,\delta_0} (\log \sigma_k^{-1})^{-2}. \quad (4.24)$$

Now we see from (4.6) and (4.20) that

$$\begin{aligned}
|\tilde{w}_k(x) + \tilde{\gamma}'_k(x) \log |x|| &\leq \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k}} \tilde{f}_k(y) |\log |y|| dy \\
&+ \frac{1}{2\pi} \left| \int_{B_{2R_0/\sigma_k} \setminus B_{|x|/2}} \tilde{f}_k(y) \log \frac{1}{|x-y|} dy \right| \\
&+ \frac{1}{2\pi} \int_{B_{|x|/2}} \tilde{f}_k(y) \left| \log \frac{|x|}{|x-y|} \right| dy + O(1) \\
&\leq \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k}} \tilde{f}_k(y) |\log |y|| dy + \frac{\log 2}{2\pi} \|\tilde{f}_k\|_{L^1(B_{2R_0/\sigma_k})} \\
&+ \frac{1}{2\pi} \left| \int_{B_{2R_0/\sigma_k} \setminus B_{|x|/2}} \tilde{f}_k(y) \log \frac{1}{|x-y|} dy \right| + O(1)
\end{aligned}$$

for  $x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}$ . Therefore, it holds that

$$|\tilde{w}_k(x) + \tilde{\gamma}'_k(x) \log |x|| = O(1) \quad \text{as } k \rightarrow \infty \quad (4.25)$$

by (4.16), (4.22) with (4.24), and the uniform  $L^1$  boundedness of  $\tilde{f}_k$ . Then (4.21) and (4.25) imply

$$\begin{aligned}
|\tilde{w}_k(x) + \tilde{\gamma}_k \log |x|| &\leq (\tilde{\gamma}_k - \tilde{\gamma}'_k(x)) \log |x| + |\tilde{w}_k(x) + \tilde{\gamma}'_k(x) \log |x|| \\
&\leq C_{9,\varepsilon_0,\delta_0} (\log \sigma_k^{-1})^{-3} \log(\sigma_k^{-1} R_0) + O(1) = O(1) \quad \text{as } k \rightarrow \infty
\end{aligned}$$

for  $x \in B_{R_0/\sigma_n} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}$ .  $\square$

Now we complete the proof of Proposition 3.

Proof of Proposition 3: We take  $\delta_0$  and  $\tilde{R}_0$  as in (4.17) and (4.18), respectively. First, (2.7) and (4.6) with (4.8) imply

$$\begin{aligned}
|\tilde{w}_k(x) + \tilde{\gamma}_k \log(1 + |x|)| &\leq |\tilde{w}_k(x)| + \tilde{\gamma}_k \log(1 + |x|) \\
&\leq C_{12}, \quad x \in B_{\tilde{R}_0},
\end{aligned} \quad (4.26)$$

while Lemma 4.5 means

$$|\tilde{w}_k(x) + \tilde{\gamma}_k \log(1 + |x|)| \leq C_{13}, \quad x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}, \quad (4.27)$$

where  $k \gg 1$ .

Now we put

$$\begin{aligned}
\tilde{w}_k^+(x) &= -\tilde{\gamma}_k \log |x| + C_{14} + \frac{C_{7,\varepsilon_0}}{9\delta_0^2} |x|^{-3\delta_0} \\
\tilde{w}_k^-(x) &= -\tilde{\gamma}_k \log |x| - C_{14} - \frac{1}{4} |x|^2 \tilde{\delta}_k \sup_{B_{2R_0}} e^{\xi_k}
\end{aligned}$$

for  $C_{14} = 1 + \max\{C_{12}, C_{13}\}$  and  $k \gg 1$ , recalling (2.6), and let

$$A_k = B_{(\log \sigma_k^{-1})^{1/\delta_0}} \setminus \bar{B}_{\tilde{R}_0}.$$

Then (4.19) implies

$$\begin{aligned} -\Delta \tilde{w}_k^+ &= C_{7,\varepsilon_0} |x|^{-(2+3\delta_0)} \geq \tilde{f}_k - \tilde{\delta}_k e^{\xi_k} \quad \text{in } A_k \\ \tilde{w}_k^+ &\geq \tilde{w}_k \quad \text{on } \partial A_k. \end{aligned}$$

Next, we have

$$\begin{aligned} -\Delta \tilde{w}_k^- &= -\tilde{\delta}_k \sup_{B_{2R_0}} e^{\xi_k} \leq \tilde{f}_k - \tilde{\delta}_k e^{\xi_k} \quad \text{in } A_k \\ \tilde{w}_k^- &\leq \tilde{w}_k \quad \text{on } \partial A_k. \end{aligned}$$

Since  $-\Delta \tilde{w}_k = \tilde{f}_k - \tilde{\delta}_k e^{\xi_k}$  in  $A_k$ , it follows from the maximum principle that

$$\tilde{w}_k^- \leq \tilde{w} \leq \tilde{w}_k^+ \quad \text{in } A_k. \quad (4.28)$$

Using

$$\left| \frac{1}{4} |x|^2 \tilde{\delta}_k \right| \leq C_{15}, \quad x \in B_{R_0/\sigma_k}$$

and

$$\left| \frac{C_{7,\varepsilon_0}}{9\delta_0^2} |x|^{-3\delta_0} \right| \leq C_{16}, \quad x \in A_k,$$

we obtain

$$|\tilde{w}_k(x) + \tilde{\gamma}_k \log |x|| \leq C_{14} + \max\{C_{15}, C_{16}\}, \quad x \in A_k \quad (4.29)$$

for  $k \gg 1$ .

Properties (4.26)-(4.29), (1.16) and (4.8) imply (1.14) for  $\alpha = 1$ ,

$$w_k(x) - w_k(0) = - \left( \frac{4}{\int_{I_+} \beta \mathcal{P}(d\beta)} + o(1) \right) \log(1 + e^{w_k(0)/2} |x|) + O(1).$$

The other case of  $\alpha$  follows from the relation  $(w_{k,\alpha}(x) - w_{k,\alpha}(0)) = \alpha(w_k(x) - w_k(0))$ , and the proof is complete.  $\square$

## 5 Proof of Theorem 1

We begin with the following lemma.

**Lemma 5.1.** *It holds that*

$$w_{k,\alpha}(0) = w_k(0) + O(1) \quad \text{as } k \rightarrow \infty \quad (5.1)$$

*uniformly in*  $\alpha \in [\alpha_{\min}, 1]$ .

*Proof.* By the monotonicity (2.1), we have only to show

$$w_{k,\alpha_{\min}}(0) = w_k(0) + O(1). \quad (5.2)$$

As shown in the previous section, estimate (1.14) is equivalent to

$$w_{k,\alpha}(x) - w_{k,\alpha}(0) = -\alpha \tilde{\gamma}_k \log(1 + e^{w_k(0)/2} |x|) + O(1), \quad (5.3)$$

where  $\tilde{\gamma}_k$  is as in (4.7). Since  $\alpha_{\min}\tilde{\gamma}_k \geq 2 + 3\delta_0$  for  $k \gg 1$  by (4.17), we use (5.3) to get

$$\begin{aligned} \int_{B_{R_0}} e^{w_{k,\alpha_{\min}}} &= O(1) \cdot e^{w_{k,\alpha_{\min}}(0)} \int_{B_{R_0}} (1 + e^{w_k(0)/2}|x|)^{-\alpha_{\min}\tilde{\gamma}_k} dx \\ &= O(1) \cdot e^{w_{k,\alpha_{\min}}(0)-w_k(0)} \int_{B_{R_0/\sigma_k}} (1 + |x|)^{-\alpha_{\min}\tilde{\gamma}_k} dx \\ &\leq O(1) \cdot e^{w_{k,\alpha_{\min}}(0)-w_k(0)} \int_{\mathbf{R}^2} (1 + |x|)^{-(2+3\delta_0)} dx. \end{aligned} \quad (5.4)$$

If (5.2) fails then (5.4) and Lemma 4.1 imply that  $w_{k,\alpha_{\min}} \rightarrow -\infty$  uniformly in  $\Omega \setminus B_{R_0/2}$ , and therefore we conclude  $\int_{\Omega} e^{w_{k,\alpha_{\min}}} \rightarrow 0$  as  $k \rightarrow \infty$ , which contradicts  $\int_{\Omega} e^{w_{k,\alpha_{\min}}} = 1$ .  $\square$

**Lemma 5.2.** *It holds that*

$$\limsup_{k \rightarrow \infty} \int_{I_+} (\bar{w}_{k,\alpha} + w_{k,\alpha}(0)) \mathcal{P}(d\alpha) > -\infty. \quad (5.5)$$

*Proof.* Fix  $x' \in \partial B_{R_0/2}$ . Then it holds that

$$\begin{aligned} \bar{w}_{k,\alpha} &= w_{k,\alpha}(x') + O(1) = w_{k,\alpha}(0) - \alpha\tilde{\gamma}_k \log(1 + e^{w_k(0)/2}|x'|) + O(1) \\ &= \left(1 - \frac{\alpha\tilde{\gamma}_k}{2}\right) w_k(0) + O(1) \end{aligned} \quad (5.6)$$

by (4.1)-(4.2), (5.3) and (5.1). Since  $\tilde{\gamma} \geq \tilde{\gamma}_k$  and  $\int_{I_+} (1 - \alpha\tilde{\gamma}/4) \mathcal{P}(d\alpha) = 0$  by (4.9) and (1.16), respectively, it follows that

$$\begin{aligned} \int_{I_+} (\bar{w}_{k,\alpha} + w_{k,\alpha}(0)) \mathcal{P}(d\alpha) &= 2w_k(0) \int_{I_+} \left(1 - \frac{\alpha\tilde{\gamma}_k}{4}\right) \mathcal{P}(d\alpha) + O(1) \\ &= 2w_k(0) \int_{I_+} \left(1 - \frac{\alpha\tilde{\gamma}}{4}\right) \mathcal{P}(d\alpha) + \frac{1}{2}w_k(0)(\tilde{\gamma} - \tilde{\gamma}_k) \int_{I_+} \alpha \mathcal{P}(d\alpha) + O(1) \\ &\geq 2w_k(0) \int_{I_+} \left(1 - \frac{\alpha\tilde{\gamma}}{4}\right) \mathcal{P}(d\alpha) + O(1) = O(1), \end{aligned}$$

where we have used (5.6) and (5.1) in the first equality.  $\square$

**Lemma 5.3.** *It holds that*

$$J_{\lambda_k}(v_k) = \frac{\lambda_k}{2} \int_{I_+} \left( \bar{w}_{k,\alpha} + \int_{\Omega} w_{k,\alpha} e^{w_{k,\alpha}} \right) \mathcal{P}(d\alpha). \quad (5.7)$$

*Proof.* By (1.11) and  $\int_{\Omega} v_k = 0$ , we have

$$\frac{1}{2} \int_{\Omega} |\nabla v_k|^2 = \frac{1}{2\alpha^2} \int_{\Omega} |\nabla w_{k,\alpha}|^2 \quad (5.8)$$

and

$$\bar{w}_{k,\alpha} = -\log \left( \int_{\Omega} e^{\alpha v_k} dx \right), \quad (5.9)$$

respectively. Multiplying (1.12) by  $w_{k,\alpha}$  and using

$$\int_{\Omega} v_k = 0, \quad w_{k,\alpha} = \alpha v_k + \bar{w}_{k,\alpha}, \quad \int_{\Omega} e^{w_{k,\beta}} = 1,$$

we get

$$\begin{aligned} \int_{\Omega} |\nabla w_{k,\alpha}|^2 &= \alpha \lambda_k \int_{\Omega} w_{k,\alpha} \left( \int_{I_+} \beta \left( e^{w_{k,\beta}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\beta) \right) dx \\ &= \alpha^2 \lambda_k \int_{\Omega} v_k \int_{I_+} \beta e^{w_{k,\beta}} \mathcal{P}(d\beta) dx \\ &= \alpha^2 \lambda_k \int_{I_+} \int_{\Omega} w_{k,\beta} e^{w_{k,\beta}} dx \mathcal{P}(d\beta) - \alpha^2 \lambda_k \int_{I_+} \bar{w}_{k,\beta} \mathcal{P}(d\beta). \end{aligned} \quad (5.10)$$

We combine (5.8)-(5.10) with  $\mathcal{P}(I_+) = 1$  to obtain

$$\begin{aligned} J_{\lambda_k}(v_k) &= \frac{1}{2} \int_{\Omega} |\nabla v_k|^2 - \lambda_k \int_{I_+} \log \left( \int_{\Omega} e^{\alpha v_k} \right) \mathcal{P}(d\alpha) \\ &= \frac{1}{2} \int_{I_+} \frac{1}{\alpha^2} \int_{\Omega} |\nabla w_{k,\alpha}|^2 dx \mathcal{P}(d\alpha) + \lambda_k \int_{I_+} \bar{w}_{k,\alpha} \mathcal{P}(d\alpha) \\ &= \frac{\lambda_k}{2} \int_{I_+} \int_{\Omega} w_{k,\alpha} e^{w_{k,\alpha}} dx \mathcal{P}(d\alpha) - \frac{\lambda_k}{2} \int_{I_+} \bar{w}_{k,\alpha} \mathcal{P}(d\alpha) \\ &\quad + \lambda_k \int_{I_+} \bar{w}_{k,\alpha} \mathcal{P}(d\alpha) = \frac{\lambda_k}{2} \int_{I_+} \left( \bar{w}_{k,\alpha} + \int_{\Omega} w_{k,\alpha} e^{w_{k,\alpha}} \right) \mathcal{P}(d\alpha). \end{aligned}$$

The proof is complete.  $\square$

We now prove Theorem 1 in the following.

*Proof of Theorem 1:* We shall show that (1.7) holds. To this end we apply  $\int_{\Omega} e^{w_{k,\alpha}} = 1$  in (5.7) and get

$$\begin{aligned} J_{\lambda_k}(v_k) &= \frac{\lambda_k}{2} \left\{ \int_{I_+} (\bar{w}_{k,\alpha} + w_{k,\alpha}(0)) \mathcal{P}(d\alpha) \right. \\ &\quad \left. + \int_{I_+} \mathcal{P}(d\alpha) \int_{\Omega} (w_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{w_{k,\alpha}(x)} dx \right\}. \end{aligned}$$

Hence the proof of (1.7) is reduced to showing

$$\int_{I_+} \mathcal{P}(d\alpha) \int_{\Omega} (w_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{w_{k,\alpha}(x)} dx = O(1), \quad (5.11)$$

thanks to (5.5).

To show (5.11), we take  $x' \in \Psi_k^{-1}(\partial B_{R_0/2})$ . Then, (4.1)-(4.2), (5.3), (5.1), (3.54) and (4.8) imply, uniformly in  $x \in \Omega \setminus \Psi_k^{-1}(B_{R_0})$  and  $\alpha \in [\alpha_{\min}, 1]$ , that

$$\begin{aligned} (w_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{w_{k,\alpha}(x)} &= e^{O(1)} (O(1) + w_{k,\alpha}(x') - w_{k,\alpha}(0)) e^{w_{k,\alpha}(x')} \\ &= e^{O(1)} \left( O(1) - \alpha \tilde{\gamma}_k \log \left( 1 + e^{w_k(0)/2} |x'| \right) \right) e^{w_{k,\alpha}(0) - \alpha \tilde{\gamma}_k \log(1 + e^{w_k(0)/2} |x'|)} \\ &= e^{O(1)} \left( O(1) - \frac{\alpha \tilde{\gamma}_k}{2} w_k(0) \right) e^{-\left( \frac{\alpha \tilde{\gamma}_k}{2} - 1 \right) w_k(0)} = o(1). \end{aligned}$$

Hence it follows that

$$\int_{[\alpha_{\min}, 1]} \mathcal{P}(d\alpha) \int_{\Omega \setminus \Psi_k^{-1}(B_{R_0})} (w_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{w_{k,\alpha}(x)} dx = o(1). \quad (5.12)$$

Finally, (5.3), (5.1), (3.54) and (4.8) imply

$$\begin{aligned} & \int_{B_{R_0}} (w_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{w_{k,\alpha}(x) + \xi_k(x)} dx \\ &= -e^{w_{k,\alpha}(0) + O(1)} \int_{B_{R_0}} e^{(w_{k,\alpha}(x) - w_{k,\alpha}(0)) + \xi_k(x)} \log(1 + e^{w_k(0)/2}|x|) dx + O(1) \\ &= -e^{w_{k,\alpha}(0) + O(1)} \int_{B_{R_0}} \frac{\log(1 + e^{w_k(0)/2}|x|)}{(1 + e^{w_k(0)/2}|x|)^{\alpha \tilde{\gamma}_k}} dx + O(1) \\ &= -e^{w_{k,\alpha}(0) - w_k(0) + O(1)} \int_{B_{e^{w_k(0)/2} R_0}} \frac{\log(1 + |x|)}{(1 + |x|)^{\alpha \tilde{\gamma}_k}} dx + O(1) = O(1) \end{aligned} \quad (5.13)$$

uniformly in  $\alpha \in [\alpha_{\min}, 1]$ . Then (5.11) follows from (5.12) and (5.13).  $\square$

## A Proof of Lemma 2.3

Given  $K > 0$ , we put

$$\begin{aligned} I_1(x) &= \int_{D_1} \frac{\log|x-y| - \log(1+|y|) - \log|x|}{\log|x|} f(y) dy \\ I_{2,K}(x) &= \int_{D_{2,K}} \frac{\log|x-y| - \log(1+|y|) - \log|x|}{\log|x|} f(y) dy \\ I_{3,K}(x) &= \int_{D_{3,K}} \frac{\log|x-y| - \log(1+|y|) - \log|x|}{\log|x|} f(y) dy, \end{aligned}$$

where

$$\begin{aligned} D_1 &= D_1(x) \equiv \{y \in \mathbf{R}^2 \mid |y-x| \leq 1\} \\ D_{2,K} &= D_{2,K}(x) \equiv \{y \in \mathbf{R}^2 \mid |y-x| > 1, |y| \leq K\} \\ D_{3,K} &= D_{3,K}(x) \equiv \{y \in \mathbf{R}^2 \mid |y-x| > 1, |y| > K\}. \end{aligned}$$

Then it holds that

$$\frac{z(x)}{\log|x|} - \gamma = \frac{1}{2\pi} (I_1(x) + I_{2,K}(x) + I_{3,K}(x)).$$

We have only to show that each  $\varepsilon > 0$  admits  $K_\varepsilon$  and  $L_\varepsilon > 0$  such that

$$|I_1(x)| + |I_{2,K_\varepsilon}(x)| + |I_{3,K_\varepsilon}(x)| \leq \varepsilon \quad (A.1)$$

for all  $x \in \mathbf{R}^2 \setminus B_{L_\varepsilon}$ .

Since

$$\frac{\log(1+|y|) + \log|x|}{\log|x|} \leq \frac{\log(2+|x|) + \log|x|}{\log|x|} \leq 3, \quad x \in \mathbf{R}^2 \setminus B_2, \quad y \in D_1(x),$$

we have

$$\begin{aligned} |I_1(x)| &\leq 3 \int_{D_1(x)} f(y) dy - \frac{1}{\log|x|} \int_{D_1(x)} f(y) \log|x-y| dy \\ &\leq 3 \int_{D_1(x)} f(y) dy - \frac{\|f\|_\infty}{\log|x|} \int_{B_1} \log|y| dy \rightarrow 0 \end{aligned} \quad (\text{A.2})$$

uniformly as  $|x| \rightarrow +\infty$ , recalling  $f \in L^1 \cap L^\infty(\mathbf{R}^2)$ .

Next, we have

$$\left| \frac{\log|x-y| - \log(1+|y|) - \log|x|}{\log|x|} \right| \leq \frac{1}{\log|x|} \left\{ \log(1+K) + \left| \log \frac{|x-y|}{|x|} \right| \right\}$$

for  $x \in \mathbf{R}^2 \setminus B_2$  and  $y \in D_{2,K}(x)$ , and thus

$$|I_{2,K}(x)| \leq \frac{1}{\log|x|} \int_{D_{2,K}(x)} \left\{ \log(1+K) + \left| \log \frac{|x-y|}{|x|} \right| \right\} f(y) dy \quad (\text{A.3})$$

for  $x \in \mathbf{R}^2 \setminus B_2$ . From

$$\frac{1}{2+|x|} \leq \frac{|x-y|}{1+|y|} \leq 1+|x|, \quad x \in \mathbf{R}^2, \quad |y-x| \geq 1,$$

we derive

$$\left| \frac{\log|x-y| - \log(1+|y|) - \log|x|}{\log|x|} \right| \leq 3, \quad x \in \mathbf{R}^2 \setminus B_2, \quad |y-x| \geq 1$$

to obtain

$$|I_{3,K}(x)| \leq 3 \int_{D_{3,K}(x)} f(y) dy \leq 3 \int_{\mathbf{R}^2 \setminus B_K} f(y) dy \quad (\text{A.4})$$

for  $x \in \mathbf{R}^2 \setminus B_2$ .

Recalling  $0 \leq f \in L^1(\mathbf{R}^2)$ , let  $\varepsilon_0 > 0$  be given. From (A.4), there exists  $K_0 > 0$  such that

$$|I_{3,K}(x)| \leq \varepsilon_0$$

for all  $K \geq K_0$  and  $x \in \mathbf{R}^2 \setminus B_2$ . Next, by (A.3) any  $K > 0$  admits  $L_K > 0$  such that

$$|I_{2,K}(x)| \leq \varepsilon_0$$

for all  $x \in \mathbf{R}^2 \setminus B_{L_K}$ , and therefore

$$|I_{2,K_0}(x)| + |I_{3,K_0}(x)| \leq 2\varepsilon_0 \quad (\text{A.5})$$

for all  $x \in \mathbf{R}^2 \setminus B_{L_{K_0}}$ .

Thus we obtain (A.1) by (A.2) and (A.5).  $\square$

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